TWO PARAMETER INTEGRAL METHODS IN LAMINAR BOUNDARY LAYER THEORY

William Macrae Lister

A Thesis Submitted for the Degree of PhD
at the
University of St Andrews

1971

# TWO PARAMETER INTEGRAL METHODS

# IN

# LAMINAR BOUNDARY LAYER THEORY

William Macrae Lister

Thesis submitted for the

Degree of Doctor of Philosophy

of the University of St. Andrews

Tm 5874

CERTIFICATE

I certify that William M. Lister has satisfied the conditions of the Ordinance and Regulations and is thus qualified to submit the accompanying application for the degree of Doctor of Philosophy.

## DECLARATION

I declare that the following thesis is a record of research work carried out by me, that the thesis is my own composition, and that it has not been presented in application for a higher degree previously.

## Abstract

The work of this thesis is concerned with the investigation and attempted improvement of an integral method for solving the two dimensional, incompressible laminar boundary layer equations of fluid dynamics. The method which is based on a theoretical two parameter representation of well known boundary layer properties was first produced by Professor S. N. Curle. Its range of application, reliability and accuracy rely on four universal functions which have been derived from known exact solutions to the boundary layer equations, and are given tabulated in terms of a pressure gradient parameter $\lambda$. This thesis seeks to improve these properties by making adjustments to the tabulated functions and also considers the extension of the method to certain compressible boundary layer problems.

The first chapter contains the development of, and background to the method and gives a critical assessment of the existing functions. This analysis indicates that the method may be improved by supplying more data for certain ranges of $\lambda$ from which the functions may be calculated; by improving the fitting process; and by the provision for small values of $\lambda$ of an analytic form for a shape parameter H which the method involves.

To supply more data two new solutions for the flows $u_1 = u_0(1+\xi)$ and $u_1 = u_0(\xi+\xi^3)$, where $\xi$ is a non-dimensional co-ordinate in the direction of the flow, are investigated. The resulting work

produces some interesting examples of the use of series expansions in boundary layer theory and these, and the results produced, are given in the second chapter.

The fitting of the functions is carried out in chapter three. Polynomial models in terms of $\lambda$ are fitted by least squares techniques to data from seven solutions and are adjusted to ensure an analytic form for H for small values of $\lambda$. A comparison of results using new and old tables indicates that an improvement has been made.

The transformation relating certain compressible and incompressible flows is next examined and the extension of the method to such problems considered. An idea due to Stewartson for assessing the relative accuracies of methods under such circumstances indicates that the method should be highly accurate, a result confirmed by the calculation of the compressible flow $u_1 = u_0(1-\xi)$ at a leading edge Mach number of four.

The thesis is concluded with a review of the work carried out and the results obtained.

## ACKNOWLEDGMENTS

I should like to express my thanks to my supervisor, Professor S. N. Curle, for his guidance and encouragement during the work contained in this thesis, to Mrs. Cunningham for her preparation of the typescript, and to Mr. McQueen for carrying out the printing of this thesis.

I am also grateful to the Science Research Council for the receipt of a Research Studentship for the period 1967-70 while I was a research student in the Department of Applied Mathematics at St. Andrews University.

CONTENTS

NOTATION: Unless defined otherwise in the text, the following notation will be used

$u$ = component of velocity within boundary layer parallel to surface

$v$ = component of velocity within boundary layer perpendicular to surface

$x$ = co-ordinate along surface

$y$ = co-ordinate perpendicular to surface

$c$ = length characteristic of distance parallel to surface

$\delta$ = length characteristic of distance perpendicular to surface

$u_1$ = external velocity at the edge of the boundary layer

$\nu$ = coefficient of kinematic viscosity

$\gamma$ = ratio of specific heats

## Chapter I - Section I: Introductory Remarks

The work of this thesis is concerned with the investigation and attempted improvement of a practical method for solving the steady, two dimensional incompressible laminar boundary layer equations of fluid dynamics. For the purposes of this thesis it will be assumed that the boundary layer forms at a fixed impermeable surface so that, in the usual notation, the underlying problem is to solve the equations of continuity and momentum which are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_1\frac{du_1}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

respectively, subject to the boundary conditions

$$u(x,0) = v(x,0) = 0;\ u(x,y) \rightarrow u_1(x) \text{ as } y \rightarrow \infty .$$

For many years the solution of these equations for a general external velocity distribution $u_1(x)$ has proved to be a very difficult problem and only with a good deal of trouble have solutions for the simplest forms for $u_1(x)$, been found. It has therefore been necessary to develop methods which solve these equations (or suitably transformed versions) approximately, the nature of the approximation coming sometimes from physical insight, and sometimes from mathematical intuition. Many such methods have been developed possessing varying degrees of accuracy, reliability and ease of application. This thesis is concerned with the investigation of one of these methods, a method based on a sound approximation and easy to apply.

It must be admitted however that the advent of the high speed computer and the resulting development of the techniques of numerical analysis have helped to make the problem of solving the basic equations more tractable (a recent review article by Smith [18] outlines some of the better methods). This, in its turn, has raised the question as to whether there is still any need for the approximate method. Provided we think of the roles of the approximate method (which may well make use of a computer) and the 'computer' method as complementary, the answer to this is yes.

There are occasions when a full computer solution to the above equations is unnecessary and where the results given by an approximate method, perhaps slightly less accurate, are of sufficient accuracy for practical purposes. This situation can often occur in the calculation of the more general properties of the boundary layer, such as the skin friction and the estimates of thickness. Also when the solution of these equations is only the first step in the solution of a related set of equations (as can happen in extensions to compressible boundary layer problems) then the problems of storing all the information might arise, even with modern machines. Here an approximate solution might be most useful in providing quickly the starting data on which the computer might then improve. Therefore so long as an approximate method does provide reliable answers quickly and efficiently its existence alongside the sophisticated (and highly accurate) computer techniques may be justified.

When what we want is a method of calculating the downstream development of quantities such as the skin friction, $\tau_w$, (especially important since it vanishes at separation) the displacement thickness $\delta_1$ and the momentum thickness $\delta_2$ it is convenient first to find an equation which relates these directly. Such an equation is simply obtained by integrating the momentum equation across the layer and substituting from the continuity equation to eliminate the normal component of velocity. The equation so obtained is called the momentum integral equation and is

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\,(u_1^2\delta_2) + u_1\,\frac{du_1}{dx}\,\delta_1$$

On defining a pressure gradient parameter $\lambda = u_1' \dfrac{\delta_2^2}{\nu}$, a shape parameter $H = \dfrac{\delta_1}{\delta_2}$, a skin friction parameter $l = \dfrac{\delta_2 \tau_w}{\mu u_1}$ and $L = 2\left\{l - \lambda\,(H+2)\right\}$ the above equation may be rewritten as

$$\frac{d}{dx}\left(\frac{\lambda}{u_1'}\right) = \frac{L}{u_1}$$

The solution of this last equation is at the crux of many methods for calculating laminar boundary layers. One approach to solving this equation, which has been made the basis for several methods, has been to make the approximation that the parameters H, L and l depend only on $\lambda$, the pressure gradient parameter. Of this type one of the most popular methods is that due to Thwaites, which shall be described in greater detail later in this chapter. From an examination of the then known solutions Thwaites [23] showed that the parameter L was almost linearly dependent on the parameter $\lambda$. By taking L to be linear in $\lambda$

he showed that the momentum integral equation could be integrated directly to yield a simple formula for the momentum thickness $\delta_2$ in terms of the external velocity distribution $u_1(x)$. With this approximation and tables of the parameters H and l versus $\lambda$, compiled using the known solutions, he produced a method, easy to apply and of considerable accuracy.

The method of Thwaites was further refined by Curle [2] who showed theoretically that a more accurate two parameter representation of L and $l^2$ is

$$L = F_o(\lambda) - \mu G_o(\lambda)$$

$$\text{and} \quad l^2 = F_1(\lambda) - \mu G_1(\lambda)$$

where $\mu = \lambda^2 \dfrac{u_1 u_1''}{(u_1')^2}$ . A careful examination of the then available range of known solutions enabled the four functions $F_o$, $G_o$, $F_1$ and $G_1$ to be tabulated against $\lambda$ . With these tables the method of Curle, described in some detail later in this chapter, produced answers to the standard problems of incompressible boundary layer theory in even better agreement with the exact solutions, than those given by the method of Thwaites

The work of this thesis is built round an investigation of these functions $F_o$, $G_o$, $F_1$ and $G_1$ as it can be seen that the usefulness of this approximation is very closely dependent on the accuracy with which these tables can be constructed. Later in this chapter the existing tables are examined and the problems concerning the range over which they may be applied with some reliability are examined. From this examination it is found that for certain ranges of $\lambda$ insufficient data has been available for an accurate determination of these tables. To remedy this, solutions to the incompressible flows with external velocity distributions $u_1 = u_o(1+\xi)$ and $u_1 = u_o(\xi+\xi^3)$, with $\xi = \frac{x}{c}$, (which cover the ranges over which insufficient data was available) have been calculated and the methods used and data calculated are presented in Chapter II. It is also found that the methods used for fitting the data may also be improved. In the third chapter these techniques are investigated and the new tables produced using better fitting techniques on a wider range of data.

Another problem which is tackled in Chapter IV is the extension of the approximate method to the solution of certain compressible boundary layer problems which may be transformed into associated incompressible flows. It is shown that these transformed flows often behave in a manner for which the standard approximate methods of boundary layer calculation are inapplicable and yield widely inaccurate results. For one such compressible flow for which an exact solution is known the calculation using the Curle approximate method is carried out. The result obtained is found to be in good agreement with the exact solution which indicates that the new method has a wider range of application than many of the existing ones.

This introductory section may be summarised by saying that the aim of this thesis is to make improvements to the method which was suggested by Curle [2]. The improvements come mainly in the way of making minor alterations to the tabulated functions $F_o$, $G_o$, $F_1$ and $G_1$ which appear in the method and on the accuracy of which the method relies. An added aim is to make the method applicable to certain types of compressible boundary layer problems and generally to improve on the range of accuracy and applicability of this method.

# Section II: Derivation of and Background to the Method

## II.1 The Governing Equations

For clarity I shall derive or quote now certain forms of the equations on which the method is based. The boundary layer equations for steady, two-dimensional incompressible flow may be written, in the usual notation, as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_1 u_1' + \nu\frac{\partial^2 u}{\partial y^2} \tag{2}$$

After integrating equation (2) with respect to y from y = 0 to y = ∞ we obtain the momentum integral equation, for zero transverse velocity at a solid surface.

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\left(u_1^2\delta_2\right) + u_1\frac{du_1}{dx}\delta_1 \tag{3}$$

This equation expresses conservation of momentum as a whole and may be written alternatively as

$$\frac{d}{dx}\left(\frac{\lambda}{u_1'}\right) = \frac{L}{u_1} \tag{4}$$

where $\lambda = \frac{\delta_2^2}{\nu} u_1'$ , a pressure gradient parameter,

$l = \frac{\delta_2}{u_1}\left(\frac{\partial u}{\partial y}\right)_w$ , a skin friction parameter,

$H = \frac{\delta_1}{\delta_2}$ , a shape parameter, and

$L = 2\left\{l - \lambda(H+2)\right\}$

A further useful integral form of the boundary layer equation may be obtained by first multiplying equation (2) through by u and then integrating from $y = 0$ to $y = \infty$. This gives

$$\frac{d}{dx}\left(\tfrac{1}{2}\rho u_1^3 \delta_3\right) = \mu \int_0^\infty \left(\frac{\partial u}{\partial y}\right)^2 dy. \quad \nu = \frac{\mu}{\rho} \tag{5}$$

and expresses the physical fact that the rate of change of the flux of kinetic energy defect within the boundary layer is equal to the rate at which kinetic energy is dissipated by viscosity and will be referred to as the kinetic-energy integral equation. Equation (5) may be rewritten as

$$\frac{d}{dx}\left(u_1^3 \delta_3\right) = \frac{2D}{\rho} \tag{6}$$

where $$D = \mu \int_0^\infty \left(\frac{\partial u}{\partial y}\right)^2 dy$$

By introducing the non-dimensional quantities $H_1 = \frac{\delta_3}{\delta_2}$; $D_1 = \frac{\delta_2 D}{\mu u_1{}^2}$ equation (6) may be rewritten as

$$\frac{d}{dx}\left(u_1^3 H_1 \delta_2\right) = \frac{2\nu u_1{}^2 D_1}{\delta_2} \tag{7}$$

If we now multiply through equation (7) by $2u_1{}^3 H_1 \delta_2$ and integrate we obtain $u_1{}^6 H_1{}^2 \delta_2{}^2 = 4\nu \int_0^x H_1 D_1 u_1{}^5 dx$

$$\text{or } \delta_2^2 = \frac{4\nu}{H_1^2} u_1{}^{-6} \int_0^x H_1 D_1 u_1{}^5 dx \tag{8}$$

At this stage it is interesting to note the circumstances under which the boundary layer equation may be integrated to give a form in which the normal velocity v does not appear. The crux of the matter is that if equation (2) is multiplied by say P and then integrated from $y = 0$ to $y = \infty$, v can be eliminated provided that $P(\frac{\partial u}{\partial y})$ is an exact differential coefficient of u and its derivatives. It is therefore clear that two families of possibilities are $P = u^n$ and $P = (\frac{\partial u}{\partial y})^m(\frac{\partial^2 u}{\partial y^2})$. The cases $n = 0$ and $n = 1$ correspond to the momentum and kinetic energy integral equations respectively. It will be seen at a later stage how use is made of these forms and how the other family $P = (\frac{\partial u}{\partial y})^m(\frac{\partial^2 u}{\partial y^2})$ also contributes useful integral forms.

## II.2 The Background to, and Development of the Method

For many years after Prandtl introduced the concept of the boundary layer the number of accurate solutions to the equations which governed its behaviour even in the simplest case of steady, two-dimensional incompressible flow were few. Of these few solutions there were two types, which will be referred to later as the 'exact' or 'special' solutions.

The first type was for those flows where the geometrical picture is very simple, such as the problem of flow past a flat plate held parallel to the stream, or where the domain considered is very limited, for example, flow sufficiently near to the stagnation point of a bluff body where the body may in effect be regarded as a plane normal to the stream. In these cases, appropriate transformations could be made to reduce the boundary layer equations to a single ordinary differential equation, the solution of this equation could then in theory be obtained to any required accuracy.

The second type of solution arose when the shape of the body is assumed such that the external velocity $u_1(x)$ may be taken as a power series in x containing say, two or three terms. In these cases, the velocity u in the boundary layer may be expanded as a power series in x, the coefficients being functions of say $y/\delta$, where $\delta$ is representative of the scale normal to the surface. These functions are obtained by solution of the appropriate sequence of ordinary differential equations of which all but the first are linear.

To obtain solutions by these methods, even for the simplest cases took a considerable amount of time and effort and attempted an accuracy which was often unnecessary for practical purposes. Much research time was therefore devoted to constructing methods which would be quicker to apply and would give acceptable if perhaps slightly less accurate solutions.

Many of the first approximate methods were based on an idea due to Pohlhausen[15]. It consisted in making some plausible assumption about the general shape of the velocity profile within the layer. In the simplest form of the method the velocity profile was taken to be a function of the non-dimensional co-ordinate normal to the wall $\eta = y/\delta$, where $\delta$ is a length characteristic of the thickness of the layer. This profile was then made to satisfy some of the same conditions as the true velocity profile did at the wall and at the edge of the boundary layer. The quantity $\delta$ was left to be determined so that the resulting u satisfied the momentum integral equation. This form of the momentum integral equation now contained quantities which could be tabulated numerically using the results from the 'special' solutions mentioned earlier. Given these tables the momentum integral equation could be integrated to yield the required quantities $\delta_1$, $\delta_2$ and $\tau_w$. The choice of the form of the approximate velocity profile and of the boundary conditions which it was made to satisfy was rather arbitrary, and often a point of contention. Pohlhausen's particular choice of velocity profile was

$$\frac{u}{u_1} = a_o + a_1\eta + a_2\eta^2 + a_3\eta^3 + a_4\eta^4$$

the coefficients being chosen to satisfy the conditions

$$u = 0,\ \frac{\partial^2 u}{\partial y^2} = -\frac{u_1}{\nu}\frac{du_1}{dx}\ ,\ \text{when } y = 0$$

and

$$u = u_1,\ \frac{\partial u}{\partial y} = 0,\ \frac{\partial^2 u}{\partial y^2} = 0 \text{ when } y = \delta$$

The method appeared in general to give good results in regions of favourable pressure gradients but was somewhat inaccurate as regards the prediction of separation. Various attempts were made to improve the method. For example, Howarth[10], based his velocity profile on the solution to the problem with external flow $u_1 = u_o(1 - \frac{x}{c})$, Walz[28] on the Falkner-Skan similarity solutions for external flows of the form $u_1 = u_o \xi^n$, and Ulrich and Schlichting[17] chose a sixth degree polynomial (in $\eta$) so that it could satisfy additional conditions at the wall as well as at the outer edge of the boundary layer. In these methods a form had to be assumed for the boundary layer velocity profile and dependence was assumed solely on the non-dimensional normal co-ordinate. This, in turn, meant dependence on the one parameter $\lambda = \frac{\delta_2^{\,2}}{\nu}\frac{du_1}{dx}$, which makes its appearance through the condition that the momentum equation should be satisfied at the wall.

The approach of replacing the velocity profile by some polynomial approximation was regarded as the most satisfactory practical approach until Thwaites[23] pointed out that if what was required was the calculation of $\delta_1$, $\delta_2$ and $\tau_w$ then a detailed knowledge of the velocity profile within the layer was not necessary, but rather all that was required was a suitable correlation between the boundary layer properties H, l, L and $\lambda$. He found that though there was some variation in the curves of H and $l v \lambda$ from solution to solution, especially for negative $\lambda$ (i.e. in regions of unfavourable pressure gradients) the variations of $L(\lambda)$ were less pronounced and that $L(\lambda)$ could be taken as roughly linear for all solutions. He found that choosing $L(\lambda) = 0.45 - 6\lambda$ gave good agreement with all the known solutions and with this form for $L(\lambda)$ equation (4) could be integrated to yield a formula for $\delta_2$ in terms of x

$$\delta_2^2 = 0.45\nu u_1^{-6}\int_0^x u_1^{\,5}dx \tag{9}$$

From the known solutions, tables were constructed of suitable values of H and l for various values of $\lambda$. The method could then be applied as follows. The quantities $\lambda$ and $\delta_2$ could be calculated immediately as functions of x using equation (9). Once these relationships were known, by making use of the tabulated values of l, the skin friction $\tau_w$ could be obtained, and using the tabulated value of H the displacement thickness $\delta_1$ could be obtained. The position of separation of the boundary layer which is taken to be where $\tau_w = 0$, was calculated by finding the value of x which gave the $\lambda$ such that $l(\lambda) = 0$. For Thwaites' method it is assumed that $l(\lambda) = 0$ when $\lambda = -0.090$. This method, essentially empirical, reproduced the known 'exact' solutions to a reasonable accuracy and with its ease of application was widely accepted as one of the better practical methods. It was shown in an even better light when Leibenson [12] and Truckenbrodt [26] showed that by making simple approximations in the kinetic energy integral equation Thwaites' fitting of L as a linear function of $\lambda$ could be justified. They pointed out that $H_1$ and $D_1$ were approximately constant over a wide range of pressure gradients. The approximate constancy of $H_1$ could be traced to the fact that it is the ratio of two integrals, each of which has an integrand which is zero at the wall. Now if we compare the velocity profiles at various stations, they will differ greatly between stagnation point and separation. But upon closer examination it is found that the slopes of the outer parts of these layers do not differ much and that the greatest changes of profile shape occur in the region close to the wall, where the velocity is small and little contribution is made to either of the integrals for $\delta_2$ and $\delta_3$. Thus $H_1 = \frac{\delta_3}{\delta_2}$ might be expected to vary but little, even though the velocity profile varies a lot. The argument for the constancy of $D_1$ is that the decrease in the contribution to the dissipation integral as the skin friction decreases to zero at separation is partially balanced by an increase in momentum thickness and a decrease in the local mainstream velocity. On making the assumption that $H_1$ and $D_1$ were constant equation (8) reduced to

$$\delta_2{}^2 = 4\nu u_1{}^{-6} \frac{D_1}{\overline{H}_1} \int_0^x u_1{}^5 \, dx \tag{10}$$

The values chosen for $H_1$ and $D_1$ were $H_1 = 1.57$, $D_1 = 0.173$ which were the values for a Blasius layer. These values as well as being rough means, are also particularly accurate in the region close to a pressure minimum where the integrand in (8) has a peak.

With these values of $D_1$ and $H_1$ equation (10) becomes

$$\delta_2{}^2 = 0.441 \nu u_1{}^{-6} \int_0^x u_1{}^5 \, dx \tag{11}$$

and $L = 0.441 - 6\lambda$, very close to Thwaites' formula. The methods considered up to this stage have the one major drawback that they all assume dependence only on the one parameter $\lambda$. Tani[21], pointed out that $\lambda$ does not exactly fix the velocity distribution and that the boundary layer characteristics especially near separation are unlikely to depend solely on the parameter $\lambda$. He mentioned that other parameters such as $\mu = \lambda^2 u_1 u_1''/(u_1')^2$ also affect the distribution, the influence becoming more marked as separation is approached. Tani did not however construct an approximate method which made use of this parameter. It is to the work of Curle[2] that we must turn to a lot. The argument for the constancy of $D_1$ is that the increase in the contribution to the dissipation integral as the skin friction decreases to zero at separation is partially balanced by an increase in momentum thickness and a decrease in the local mainstream velocity. On making the assumption that $H_1$ and $D_1$ were constant equation (4) reduced to

$$\delta_2^{\,2} = 4\nu u_1^{-6}\,\frac{D_1}{\bar{H}_1}\int_0^x u_1^{\,5}\,dx \qquad (10)$$

The values chosen for $H_1$ and $D_1$ were $H_1 = 1.57$, $D_1 = 0.173$ which were the values for a Blasius layer. These values as well as being rough means, are also particularly accurate in the region close to a pressure minimum where the integrand in (4) has a peak.

With these values of $D_1$ and $H_1$ equation (10) becomes

$$\delta_2^{\,2} = 0.441\,\nu u_1^{-6}\int_0^x u_1^{\,5}\,dx \qquad (11)$$

and $L = 0.441 - 6\lambda$, very close to Thwaites' formula. The methods considered up to this stage have the one major drawback that they all assume dependence only on the one parameter $\lambda$. Tani[21], pointed out that $\lambda$ does not exactly fix the velocity distribution and that the boundary layer characteristics especially near separation are unlikely to depend solely on the parameter $\lambda$. He mentioned that other parameters such as $\mu = \lambda^2 u_1 u_1''/(u_1')^2$ also affect the distribution, the influence becoming more marked as separation is approached. Tani did not however construct an approximate method which made use of this parameter. It is to the work of Curle[2] that we must turn to see how this new parameter, $\mu$, mentioned by Tani, was put to use.

What Curle did was as follows. In equation (8) instead of making the assumption that $H_1$ = const, $D_1$ = const he made the small correction $H_1 = H_1(\lambda)$, $D_1 = D_1(\lambda)$. After differentiating this equation with respect to x and re-arranging he deduced that

$$u_1 \frac{d}{dx}\left(\frac{\lambda}{u_1'}\right) + 6\lambda = 4\frac{D_1}{H_1} - 2\lambda\frac{u_1}{u_1'} \cdot \frac{1}{H_1}\frac{dH_1}{dx} \tag{12}$$

and by equation (4) this yields

$$L + 6\lambda = 4\frac{D_1}{H_1} - 2\lambda\frac{u_1}{u_1'} \cdot \frac{1}{H_1}\frac{dH_1}{dx} \tag{13}$$

But

$$\frac{dH_1}{dx} = \frac{dH_1}{d\lambda} \cdot \frac{d\lambda}{dx} = H_1' \left\{ L\frac{u_1'}{u_1} + \lambda\frac{u_1''}{u_1'} \right\}^* \tag{14}$$

On substituting this form into (13) we obtain

$$L + 6\lambda = 4\frac{D_1}{H_1} - 2\lambda\frac{H_1'}{H_1}\left\{ L + \lambda\frac{u_1 u_1''}{(u_1')^2} \right\} \tag{15}$$

or

$$L\left\{1 + 2\lambda\frac{H_1'}{H_1}\right\} = 4\frac{D_1}{H_1} - 6\lambda - 2\mu\frac{H_1'}{H_1} \tag{16}$$

where

$$\mu = \lambda^2 u_1 u_1''/(u_1')^2 \tag{17}$$

Equation (16) can be written in the form

$$L = F_o(\lambda) - \mu G_o(\lambda) \tag{18}$$

where
$$F_o(\lambda) = \frac{4D_1 - 6\lambda H_1}{H_1 + 2\lambda H_1'} \; ; \; G_o(\lambda) = \frac{2H_1'}{H_1 + 2\lambda H_1'}$$

*Primes denote derivatives with respect to $\lambda$, except in the case of $u_1$ where primes denote derivatives with respect to x.

Equation (18) gives a convenient form for L and involves the parameter $\mu$ which had been mentioned by Tani. The problem now reduces to trying to find universal forms for $F_o(\lambda)$ and $G_o(\lambda)$ and also to trying to determine whether it is possible to express l in a similar form. With the solutions then at his disposal ($u_1 = u_o\xi^n$, $u_1 = u_o(1 - \dot{\xi})$, $u_1 = u_o \sin\xi$ and $u_1 = u_o(1 - \xi^2)$) Curle made a careful examination of the sets ($L, \lambda, \mu$) to see whether it was possible to determine a function $G_o(\lambda)$ such that the results $L + \mu G_o(\lambda)$ when plotted against $\lambda$ all fall on to one curve. After a certain amount of intelligent trial and error $G_o(\lambda)$ was chosen such that $G_o(\lambda) = 0.66 + 3\lambda$. The new values of L were then given by $L(\lambda,\mu) = F_o(\lambda) - \mu G_o(\lambda)$ where $F_o(\lambda)$ and $G_o(\lambda)$ could be tabulated once and for all. Once the value of $L(\lambda,\mu)$ is determined, the momentum thickness $\delta_2(x)$ is given by the solution of the equation

$$u_1 \frac{d}{dx}\left(\frac{\delta_2^2}{\nu}\right) = L(\lambda,\mu) = F_o(\lambda) - \mu G_o(\lambda) \tag{19}$$

A simple method of solution of this equation is by iteration. Thus we write

$$F_o(\lambda) - \mu G_o(\lambda) = 0.45 - 6\lambda + g(\lambda,\mu) \tag{20}$$

where the term $g(\lambda,\mu)$ defined by the above equation is the correction to the Thwaites' formula. Since

$$g(\lambda,\mu) = F_o(\lambda) - 0.45 + 6\lambda - \mu G_o(\lambda) \tag{21}$$

the function $F_o(\lambda) - 0.45 + 6\lambda$ has also been tabulated. Equation (19) then becomes

$$u_1 \frac{d}{dx} \left(\frac{\delta_2^2}{\nu}\right) + 6u_1' \frac{\delta_2^2}{\nu} = 0.45 + g(\lambda, \mu)$$

which may be formally integrated to give

$$\delta_2^2 = 0.45 \nu u_1^{-6} \int_0^x (1 + 2.22g)\, u_1^5 \, dx \qquad (22)$$

For a first approximation $g(\lambda, \mu)$ is set equal to zero, and once the corresponding values of $\lambda$ and $\mu$ have been determined, these are used to estimate $g(\lambda, \mu)$ for a second approximation and so on.

To express l in a similar form was a more difficult task and led Curle to look for new integral forms of the boundary layer equations. Further members of the family $P = u^n$ produced no more useful relationships and it was to the family $P = (\frac{\partial u}{\partial y})^m (\frac{\partial^2 u}{\partial y^2})$ that he next turned. The first member $m = 0$, yielded the form

$$\frac{d}{dx} \left( \int_0^\infty u \left(\frac{\partial u}{\partial y}\right)^2 dy \right) = 2u_1 u_1' \left(\frac{\partial u}{\partial y}\right)_w - 2\nu \int_0^\infty \left(\frac{\partial^2 u}{\partial y^2}\right)^2 dy \qquad (23)$$

There are two reasons why this equation (23) did not produce a suitable form for the skin friction parameter l, and, following the arguments of Curle [2], they are.

In the above equation (23) it would be convenient to write

$$A = \frac{\delta_2}{u_1^3} \int_0^\infty u \left(\frac{\partial u}{\partial y}\right)^2 dy; \quad B = \frac{\delta_2^3}{u_1^2} \int_0^\infty \left(\frac{\partial^2 u}{\partial y^2}\right)^2 dy$$

and examine whether A and B are roughly constant. If this were so, then the improvement could be made that $A = A(\lambda)$, $B = B(\lambda)$, and then proceed exactly as before. Unfortunately it was shown by Curle that the rough values of A and B calculated for a stagnation point, Blasius layer and a typical separation profile, indicated that the values of B varied by factors of up to 3 or 4 and A by factors of up to 8 or 9, so A and B are far from constant.

Secondly, it is well known that the various curves of l against $\lambda$ drop sharply to zero as separation is approached, because of the singularity in the boundary layer equations which almost certainly exists at that point. This singularity in the boundary layer equations at separation shows up as a square root singularity in l. In view of the fact that these sharp falls in the values of l occur at different values of $\lambda$ it is difficult to see how the curves of l against $\lambda$ can be collapsed on to a single curve.

For these reasons equation (23) had to be discarded.

The next member with m = 1 yielded the form:-

$$\frac{d}{dx}\left(\int_0^\infty u\left(\frac{\partial u}{\partial y}\right)^3 dy\right) = 3u_1u_1'\left(\frac{\partial u}{\partial y}\right)_w^2 - 6\nu\int_0^\infty \frac{\partial u}{\partial y}\left(\frac{\partial^2 u}{\partial y^2}\right)^2 dy \qquad (24)$$

This is an equation for $l^2$ and since the singularity in l is square-root in nature, the curves of $l^2$ against $\lambda$ are well-behaved even close to separation. On examination of the quantities

$$C = \frac{\delta_2^2}{u_1^4}\int_0^\infty u\left(\frac{\partial u}{\partial y}\right)^3 dy,\ D = \frac{\delta_2^4}{u_1^3}\int_0^\infty \frac{\partial u}{\partial y}\left(\frac{\partial^2 u}{\partial y^2}\right)^2 dy \qquad (25)$$

it was found that they were approximately constant. Substitution of the forms of equation (25) into (24) yields

$$\frac{d}{dx}\left(C\,\frac{u_1^4}{\delta_2^2}\right) = \frac{3u_1^3u_1'}{\delta_2^2}\,l^2 - 6\nu\frac{u_1^3}{\delta_2^4}\,D \qquad (26)$$

In this equation we note that by setting $C$ = const, $D$ = const we retrieve the Thwaites' condition which is $l = l(\lambda)$. For separation this becomes $l(\lambda) = 0$. If we then take $C = C(\lambda)$ and $D = D(\lambda)$ this yields the improved formula

$$l^2 = F_1(\lambda) - \mu G_1(\lambda) \tag{28}$$

where $F_1(\lambda)$ and $G_1(\lambda)$ are rather complicated functions of $C$ and $D$. The search for a function $G_1(\lambda)$ to make the results of $l^2 + \mu G_1(\lambda)$ when plotted against $\lambda$ fall on to a single curve for all solutions was carried out in a manner similar to that for $G_o(\lambda)$. The resulting function $G_1(\lambda)$ was given as a numerical tabulation. The separation condition becomes

$$0 = F_1(\lambda) - \mu G_1(\lambda)$$

The method of completing a calculation of the usual boundary layer characteristics is now quite straightforward. Having determined $\lambda$ and $\delta_2$ as functions of $x$, as described earlier, the values of $l$ and hence $\tau_w$ are given by equation (28). Equation (20) may be used to determine $H$, whence $\delta_1 = H\delta_2$ follows.

This new method only slightly more complicated than that of Thwaites produces answers which have errors which are typically only 5% of those given by the Thwaites' method.

## Section III: Critical Assessment of the Problem

### III.1 Introduction

In the previous section the method was derived and set against the background of boundary layer theory. In this section a discussion is given of some of the ideas behind the construction of an approximate method based on integral forms of the boundary layer equation and these ideas are illustrated with reference to the Curle 2-parameter method. This discussion indicates certain points which may require further investigation. Later in the section an outline is given as to how these points may be tackled. Finally in this section an indication is given as to how the method may be extended to certain compressible boundary layer problems. An outline is given of some of the analysis from which it may be seen for what types of problem the method may be expected to give good results.

### III.2 Pinpointing the Problems

What we must first do is to clarify the necessary ingredients, and underlying ideas behind an approximate method. We have already several 'exact' solutions which cover widely differing flows and we wish to be able to make the best use of this information to predict what happens for a more general situation. If we know some underlying theoretical relationships relating quantities, involved in the individual solutions, from one solution to another, we examine our available data in this light. If we have no such guide we have to try to find an underlying form 'between solutions' empirically. These two ideas are complementary and the difference in approach may be seen by comparing the derivation of the Thwaites' method and the Curle 2-parameter method.

Thwaites examined the curves L versus $\lambda$ from all solutions (then available) and decided to fit the form $L = a + b\lambda$ to these points. This was later justified by the work of Leibenson[12] and Truckenbodt[26] who, as described earlier, showed that Thwaites' empirical form amounted to taking $H_1$ = const, $D_1$ = const in the non-dimensional form of the kinetic energy integral equation. On the other hand Curle first shows that taking $H_1 = H_1(\lambda)$, $D_1 = D_1(\lambda)$ yields a better 'between solutions' representation for L in the form

$$L = F_\circ(\lambda) - \mu G_\circ(\lambda) \text{ where } \mu = \lambda^2 \frac{u_1 u_1''}{(u_1')^2}$$

and that a similar type of approximation and more complicated integrated form of the boundary layer equation yields the form

$$l^2 = F_1(\lambda) - \mu G_1(\lambda)$$

He next proceeds to examine the available data in the light of these forms and tries to extract universal forms for $F_\circ(\lambda)$, $G_\circ(\lambda)$, $F_1(\lambda)$, $G_1(\lambda)$ to give good agreement for L and $l^2$ with all solutions. At this point we can see over what range the method may be expected to work. Our underlying forms depend on the two parameters $\lambda$ and $\mu = \lambda^2 \frac{u_1 u_1''}{(u_1')^2}$ . The range of the method is therefore determined by the range of $\lambda$ over which there are exact solutions available from which the universal functions $F_\circ(\lambda)$, $G_\circ(\lambda)$, $F_1(\lambda)$ and $G_1(\lambda)$ may be calculated.

The more solutions available over a particular range the more confident we can be that we can choose the universal functions to describe what is actually happening and remove any bias towards one solution or another. If we examine those solutions on which the current Curle tables are based we may see where an improvement may be made. At the time that the tables were derived Curle had at his disposal the following exact solutions $u_1 = u_0(1 - \xi)$, $u_1 = u_0\xi^n$, $u_1 = u_0(1 - \xi^2)$, $u_1 = u_0\sin\xi$. From an examination of the values of $\lambda$ which these solutions yield it is seen that they are heavily weighted towards the negative end of the range. The tables constructed from these solutions are reliable for the range of $\lambda$ which these solutions cover. However for problems which yield values of $\lambda$ outside the range of $\lambda$ covered by these solutions the tables possibly give less accurate results, with greatest possibility of inaccuracy being in the range $\lambda > 0.0685$ (beyond Terrill's solution for $u_1 = u_0\sin\xi$) where the only solution used is $u_1 = u_0\xi^n$. To remedy this two new exact solutions were found corresponding to flows with external velocities $u_1 = u_0(\xi + \xi^3)$ and $u_1 = u_0(1 + \xi)$. The first solution yields data for $0.0855 \leq \lambda \leq 0.0984$ and the second for $0.0 \leq \lambda \leq 0.0855$. The techniques used for these problems are outlined in the next section and dealt with at greater length in Chapter 2.

We next have to consider how we estimate the accuracy of an approximate method. This has to be done a posteriori and conventionally as follows. The 'exact' solutions are reproduced by the approximate method and the agreement between the 'approximate' and 'exact' solution examined. The better the agreement, the better the method is. In terms of the Curle method the accuracy therefore depends on how closely L and $l^2$ reproduce the known data. This in turn depends on how accurately we can determine the functions $F_o(\lambda)$, $G_o(\lambda)$, $F_1(\lambda)$ and $G_1(\lambda)$. As was described previously what Curle did was to examine a few solutions and on choosing $G_o(\lambda) = 0.66 + 3\lambda$ found that the calculated values of $L + \mu G_o(\lambda)$ fell almost on to one curve when plotted against $\lambda$. This curve was smoothed and the smoothed values were chosen as $F_o(\lambda)$. A similar analysis was performed for $l^2$ but there was no obvious linear form for $G_1(\lambda)$ which reduced all the solutions $l^2 + \mu G_1(\lambda)$ to the one curve. However $G_1$ could be found as a numerical tabulation versus $\lambda$ to make $l^2 + \mu G_1(\lambda)$ fall almost on to a smooth curve, and these values could be taken for $F_1(\lambda)$. An attempt has been made to improve the values of these numerical tables, as will be described later, mainly by using techniques of least squares curve fitting on a function of two variables.

Another problem arises in the determination of the function H. Once the functions $F_o(\lambda)$, $G_o(\lambda)$, $F_1(\lambda)$ and $G_1(\lambda)$ have been tabulated then, given any external velocity $u_1(\xi)$ we can determine $\lambda$ and $\mu$ and so obtain L and $l^2$. H is then given from the relationship $L = 2\left\{l - \lambda(H + 2)\right\}$ i.e. $H = \dfrac{2l - L}{2\lambda} - 2$. For small values of $\lambda$, H will become discontinuous unless the functions $l^2$ and L are very carefully defined to avoid rounding errors, etc. To prevent this an analytic form for H was derived which would be valid for small $\lambda$.

To conclude this section and lead on to the next we may say that there are two broad areas in which a further investigation of Curle's two parameter method may be profitable.

(1) In extending the range over which the method may be applied with some reliability.

(2) In improving the existing tables by a better fitting of the data, and in the region of small $\lambda$ producing an analytic form for H.

III.3 Extending the Range

As described in the previous section a possible improvement to the method could be to add in more solutions, for the derivation of the tables, for positive values of $\lambda$. This has been done by investigating and deriving accurate solutions for the two problems $u_1 = u_0(\xi + \xi^3)$ and $u_1 = u_0(1 + \xi)$. An outline of the methods used (principally in connection with the flow $u_1 = u_0(\xi + \xi^3)$) is given below and the details are given in Chapter 2.

Initially it was hoped to use methods similar to those used by Curle in his treatment of the flow with external velocity $u_1 = u_0(\xi - \xi^3)$. Briefly what this involves is as follows.

When the external velocity is given as a power series in $\xi$, a formal solution of the boundary layer equations may be obtained by expanding the stream function as a power series whose coefficients are functions of the distance normal to the wall. Thus for a main stream $u_1 = \sum_{0}^{\infty} a_{2n+1} \xi^{2n+1}$ the stream function $\psi$ may be written as

$$\psi = (a_1 \nu c)^{\frac{1}{2}} \sum_0^\infty \xi^{2n+1} F_{2n+1}(\eta) \text{ where } \eta = (\frac{a_1}{\nu c})^{\frac{1}{2}} y$$

It has been shown by Howarth [9] that the functions $F_{2n+1}(\eta)$ may be expressed as linear combinations of a sequence of universal functions, which can be solved and tabulated once and for all. Once these universal functions have been tabulated the solution of any boundary layer problem with an external velocity of the above form can be reduced to the arithmetic of these functions.

It follows by differentiation that the velocity u in the boundary layer is

$$u = a_1 \sum_0^\infty \xi^{2n+1} F'_{2n+1}(\eta)$$

and that the skin friction at the wall is derived from

$$(\frac{\partial u}{\partial y})_w = (\frac{a_1^3}{\nu c})^{\frac{1}{2}} \sum_0^\infty \xi^{2n+1} F''_{2n+1}(0)$$

Using the definitions similar series expansions may be derived for $\delta_1$ and $\delta_2$.

One drawback of this type of method is that it is difficult to calculate any more than the first few terms of the series expansions involved and often these fail to converge except for very small values of $\xi$. Following an idea of Howarth[10], Curle made the following assumptions about the series expansions.

(a) Assume that the convergence of the series is such that the seventh and subsequent terms may be treated as a relatively small correction to the first six. The validity of this is examined a posteriori.

(b) Assume that the contribution of the seventh and subsequent terms may be adequately estimated by assuming that the $F_{2n+1}(\eta)$, $n \geq 6$, are similar in shape to $F_{11}(\eta)$.

On the basis of these approximations we may write the velocity as

$$\frac{u}{a_1} = \sum_{0}^{4} \xi^{2n+1} F'_{2n+1}(\eta) + A(\xi) F'_{11}(\eta)$$

and the skin friction as

$$T = \left(\frac{\nu c}{a_1^3}\right)^{\frac{1}{2}} \left(\frac{\partial u}{\partial y}\right)_w = \sum_{0}^{4} \xi^{2n+1} F''_{2n+1}(0) + A(\xi) F'_{11}(0) \qquad *$$

where $A(\xi)$ is to be determined.

Using a differentiated form of the momentum equation evaluated at the wall this approximation enables the calculation of the skin friction $(T)$ to be reduced to the solution of the simple, non-linear ordinary differential equation

$$T \frac{dT}{d\xi} = P + \frac{F'_{11}(0)}{F''_{11}(0)} T$$

where P is a known polynomial, determined from a knowledge of the mainstream velocity. Once T has been obtained, the relevant value of $A(\xi)$ is deduced from (*) by subtracting off the series expansion terms.

The displacement thickness is obtained using

$$u_1 \delta_1 = \int_0^\infty (u_1 - u)\,dy \text{ and yields, in terms of } A(\xi)$$

$$\frac{u_1}{(a_1 \nu c)^{\frac{1}{2}}} \delta_1 = \lim_{\eta \to \infty} \left\{ \frac{u_1 \eta}{a_1} - \sum_{0}^{4} \xi^{2n+1} F_{2n+1}(\eta) - A(\xi) F_{11}(\eta) \right\}$$

The momentum thickness $\delta_2$ is obtained by the solution of the momentum integral equation using the expansions for $\tau_w$ and $\delta_1$.

The above technique worked well for the flow $u_1 = u_0(\xi - \xi^3)$, which, since separation occurred for $\xi = 0.655$, covered only small values of $\xi$. For the flow $u_1 = u_o(\xi + \xi^3)$ the assumption (a) is violated once $\xi \gtrsim 0.5$ and so the method is of limited usefulness. Up to this point it does however give a highly accurate prediction of what happens and so its use may be justified in giving the initial development of the solution. Originally it was hoped that the above technique would give the development of the boundary layer to larger values of $\xi$ than it was found to do. It was hoped that these values could then be matched in with an asymptotic theory which was developed as follows.

For large $\xi$, $u_{o}(\xi + \xi^3) \sim u_o \xi^3$ and it is assumed that the velocity within the layer will behave only slightly differently from the corresponding Falkner-Skan solution for the external flow $u_1 = u_o \xi^3$. Since the behaviour of these solutions is well known it was thought that it might be useful to work out the development of the solution for large values of $\xi$ in terms of a perturbation to the Falkner-Skan solution for the flow $u_1 = u_o \xi^3$. This was done by defining the stream function $\psi(\xi, \eta)$ for the flow with external velocity $u_1 = u_o(\xi + \xi^3)$ to be of the form

$$\psi(\xi, \eta) = \left(\frac{u_o \nu c}{2}\right)^{\frac{1}{2}} \xi^2 f(\eta) + G(\eta, \xi)$$

where $\xi = \frac{x}{c}$, $\eta = \left(\frac{2u_o}{\nu c}\right)^{\frac{1}{2}} \xi y$ and $f(\eta)$ satisfies the equation

$$f''' + ff'' + \frac{3}{2}(1 - f'^2) = 0 ,$$

where $f(0) = f'(0) = 0$, $f' \to 1$ as $\eta \to \infty$. To test possible forms for $G(\eta, \xi)$ use was made of the work of Libby and Chen[13] and once the form of the perturbation solution had been determined the equation governing the first term in the expansion was solved. Though, in the event, this analysis was not used to produce the relevant parameters, it gave an interesting insight into their asymptotic behaviour.

The approach to the problem which did provide satisfactory answers was as follows. For both of the flows under discussion it was observed from the work of others, that in the range of interest of $\lambda$, H, l and L were almost linear functions of $\lambda$. It was therefore decided to try to express H, l and L as functions of $\lambda$, in the hope that these new forms would give, to a reasonable accuracy, the correct limiting values corresponding to limiting values of $\lambda$. This was done by first expressing $\lambda$, H, l and L as functions of $\xi$ and then eliminating $\xi$ between $\lambda$ and H, $\lambda$ and l and $\lambda$ and L to give $H(\lambda)$, $l(\lambda)$ and $L(\lambda)$. The series expansions which resulted gave good agreement with the known limiting values and only needed the addition of small correction terms.

For the problem $u_1 = u_0(1 + \xi)$ this is the only approach discussed. Once L had been calculated as a function of $\lambda$ an attempt was made, in each case, to solve the momentum integral equation by a Runge-Kutta type method. Some success was obtained with this method over the whole range of $\xi$ for the problem $u_1 = u_0(1 + \xi)$, but unfortunately little success was met with for the flow $u_1 = u_0(\xi + \xi^3)$.

During the course of the investigation of these two problems it was found that certain slowly convergent series expansions related to the non-dimensional displacement thickness could be made to converge more rapidly under suitable transformation. These ideas are dealt with in the last section of chapter two.

## III.4 Improving the Accuracy

### III.4.1 Collection of Data

To improve the accuracy of the tables, the solutions used in calculating the original tables were found to a greater number of figures and then the forms for L and $l^2$ fitted to these by means of least squares.

The solutions used were as follows.

A. The Falkner-Skan Solutions. $u_1 = u_0 \xi^n$, $\xi = \frac{x}{c}$

The values used were those due to Evans [6], plus a few of the original Hartree [8] values.

B. The Howarth Solution. $u_1 = u_0(1 - \xi)$, $\xi = \frac{x}{c}$

The original Howarth solution[10] for this flow was tabulated to more figures to allow for rounding errors.

C. Tani's solution. $u_1 = u_0(1 - \xi^2)$, $\xi = \frac{x}{c}$

Professor Tani very kindly supplied an improved version of the table of values (appearing on page 96, Chapter 3) with the new values tabulated to four significant figures.

D. P. G. Williams solution

As one of the intentions in modifying the method was to make it of more use in tackling compressible boundary layer problems (in conjunction with the Stewartson-Illingworth transformation) it was thought to be useful to include in the data a solution for an incompressible flow associated through the Stewartson-Illingworth transform with a compressible boundary layer problem. That solution chosen was the incompressible flow associated with the compressible flow with external velocity distribution $u_1 = u_o(1-\xi)$, zero heat transfer at the wall, Prandtl number $\sigma$ = unity, the viscosity $\mu$ proportional to the absolute temperature T and Mach number at the leading edge being equal to 4. P. G. Williams of University College, London supplied me with details of his solution to this problem from which the relevant parameters could be calculated.

E and F The flows $u_1 = u_o(\xi+\xi^3)$ and $u_1 = u_o(1+\xi)$

An outline of the methods used has been given in the previous section and these will be given in more detail in Chapter II.

G. Terrill's solution. $u_1 = u_o \sin\xi$

The required parameters were calculated from data given in Terrill's paper, (reference 22).

### III.4.2 Fitting of the Data

Once values of the parameters from the solutions mentioned in the previous section had been calculated, from these a set of sixty-two points was chosen covering the range $-0.1167 \leq \lambda \leq 0.141$ and also representing all the solutions.

Since the initial approximation for $G_o(\lambda)$ was linear in $\lambda$ it was decided to attempt to fit the functions $F_o$, $G_o$, $F_1$ and $G_1$ as polynomials in $\lambda$, over the whole range. It was found that a convenient and flexible method of fitting these forms was to take the models as

$$L = a_o + a_1\lambda + a_2\lambda^2 + \dots a_n\lambda^n + b_o\mu + b_1\mu\lambda + \dots b_k\mu\lambda^k$$
$$l^2 = c_o + c_1\lambda + \dots\dots c_q\lambda^q + d_o\mu + \dots\dots d_p\mu\lambda^p$$

(where n, k, q and p had to be determined) rewrite them as

$$L = a_o + a_1X_1 + a_2X_2 \ldots a_nX_n + b_oY_o + \ldots b_rY_r$$

$$l^2 = c_o + c_1X_1 \ldots\ldots c_qX_q + d_oY_o \ldots\ldots d_pY_p$$

by defining $X_1 = \lambda^1$, $X_2 = \lambda^2$, etc. $Y_o = \mu\lambda^o$, $Y_1 = \mu\lambda^1$, etc.

and then use the IBM Scientific Subroutine package REGRE which is used in multiple linear regression. The functions $F_o(\lambda)$, $G_o(\lambda)$, $F_1(\lambda)$ and $G_1(\lambda)$ could then be determined from

$$F_o(\lambda) = \sum_{r=0}^{n} a_r\lambda^r; \; G_o(\lambda) = -\sum_{r=0}^{k} b_r\lambda^r$$

$$F_1(\lambda) = \sum_{r=0}^{q} c_r\lambda^r; \; G_1(\lambda) = -\sum_{r=0}^{p} d_r\lambda^r$$

where the negative sign is taken in $G_o(\lambda)$ and $G_1(\lambda)$ to give agreement with the approximations, originally used, $L = F_o(\lambda) - \mu G_o(\lambda)$ and $l^2 = F_1(\lambda) - \mu G_1(\lambda)$. From an examination of the flows $u_1 = u_o(1-\xi)$ and $u_1 = u_o(1+\xi)$ for which $\mu = 0$, some idea as to the orders of polynomial approximation for $F_o(\lambda)$ and $F_1(\lambda)$ was found so that a certain amount of the work in fitting these functions over the whole range could be cut down. For comparison of models the residual sum of squares, mean modulus residual and root mean square residual in each case were calculated. Once forms had been decided upon from use of the sixty-two points, these forms were tested by calculating the above-mentioned residual quantities at twenty-five points which had not been used in the fitting.

For small values of $\lambda$, $-0.020 \leq \lambda \leq 0.020$, a similar separate analysis was carried out with the added restriction that the leading coefficients in the forms for L and $l^2$ should be related such that $c_o = a_o^2/4$ and $d_o = a_ob_o/2$ to ensure that H would be continuous for $\lambda = 0$. For $\lambda$ in this range the leading terms in a form for H were also determined.

From these forms a tabulation of numerical values of these functions was compiled. These tables and the methods used to obtain them are described in greater detail in Chapter III.

III.5 Extension to certain compressible boundary layer problems

It has been shown independently by Illingworth[11] and Stewartson[19] that a compressible laminar boundary layer problem can be reduced exactly to an associated incompressible problem subject to the following conditions:

a) there is zero heat transfer at the wall

b) the Prandtl number $\sigma$ is unity

c) the viscosity $\mu$ is assumed to be directly proportional to the absolute temperature T.

However the transformed flow, especially at higher Mach numbers, is often very different from any of the flows for which existing methods produce good results. It was therefore thought to be useful to examine certain transformed flows and the application of the Curle theory to them. In Chapter IV this has been done. The effect of the Stewartson-Illingworth transformation is discussed and an outline of an idea due to Stewartson for assessing the relative accuracies of approximate methods, at high Mach number, is given. This idea is then applied to Curle's method and from the analysis it is concluded that the method is an improvement on the better existing methods.

## CHAPTER 2: The problems $u_1 = u_0(\xi + \xi^3)$ and $u_1 = u_0(1 + \xi)$, $\xi = \frac{x}{c}$

### Section I: Introduction

This chapter deals in greater detail with some of the ideas outlined in Section III of Chapter 1 for solving the problems $u_1 = u_0(\xi + \xi^3)$ and $u_1 = u_0(1 + \xi)$. Most of the analysis was carried out for the problem $u_1 = u_0(\xi + \xi^3)$ and this problem will therefore be given more attention than the problem $u_1 = u_0(1 + \xi)$ for which only that method which did produce the eventual results is quoted.

The next section deals with the Howarth-Blasius series approach to solving $u_1 = u_0(\xi + \xi^3)$ and with Curle's idea for extrapolating the values of the skin friction. Though it had been hoped that this approach could be extended to higher values of $\xi$ than it was eventually found to do, the analysis is produced as a highly accurate initial development of the flow.

The third section deals with the asymptotic theory for large values of $\xi$ when $u_0(\xi + \xi^3) \approx u_0\xi^3$, and the velocity within the layer is assumed to be only slightly different from the Falkner-Skan solution. An analysis is carried out to find the form of the co-ordinate perturbation expansion from the Falkner-Skan solution and the equation governing the first term in this series is solved. With these forms the asymptotic behaviour of the parameters can be examined and certain of the difficulties which were experienced in attempting the matching of the large and small value $\xi$ expansions are explored.

The fourth section describes the analysis which did provide satisfactory results. It involves the calculation of the parameters H, l, L, $\lambda$ as functions of $\xi$ and then the elimination of $\xi$ to give $H(\lambda)$, $l(\lambda)$, $L(\lambda)$. These expansions required only slight adjustment to give the correct asymptotic form.

In section five, the successful techniques are described for the flow $u_1 = u_0(1 + \xi)$ and section six deals with the application of Runge-Kutta type methods to the solution of the momentum integral equation once L has been given as a function of $\lambda$.

The seventh and final section contains some interesting analysis of some series expansions which have arisen in connection with the non-dimensional displacement thicknesses for the flows $u_1 = u_0(\xi + \xi^3)$ and $u_1 = u_0(1 + \xi)$. It has been found that the relevant series expansions, slowly convergent in terms of Howarth-Blasius variables can be made to converge much more quickly when recast in terms of a new variable which is found by using a suitably chosen Euler transformation.

Section II: Initial Series Expansions

As has been described in Chapter 1, where the mainstream velocity $u_1$, may be expressed in the form

$$u_1 = \sum_{0}^{\infty} a_{2n+1} \xi^{2n+1}, \quad \xi = \frac{x}{c} \tag{2.1}$$

then the stream function $\psi$ may also be expressed as an odd polynomial in $\xi$

$$\psi = (a_1\nu c)^{\frac{1}{2}}\sum_{0}^{\infty}\xi^{2n+1}\ F_{2n+1}(\eta) \tag{2.2}$$

where the coefficients are functions of the scaled co-ordinate $\eta = (\frac{a_1}{\nu c})^{\frac{1}{2}}$ y, normal to the wall. These functions, $F_{2n+1}(\eta)$, as described previously, may be expressed as linear multiples of certain universal functions (Howarth [9]) which have been calculated by various workers. The velocity within the boundary layer, u, and the non-dimensional skin friction $T = (\frac{\nu c}{a_1^3})^{\frac{1}{2}}(\frac{\partial u}{\partial y})_w$ are then given by

$$\frac{u}{a_1} = \sum_{0}^{\infty}\xi^{2n+1}\ F'_{2n+1}(\eta) \tag{2.3}$$

and

$$T = (\frac{\nu c}{a_1^3})^{\frac{1}{2}}\ (\frac{\partial u}{\partial y})_w = \sum_{0}\xi^{2n+1}\ F''_{2n+1}(0) \tag{2.4}$$

The coefficients of all terms up to and including $\xi^{11}$ have been calculated and, for small values of $\xi$, the small contribution of subsequent terms may be evaluated by assuming that their dependence upon $\eta$ is similar to that of the coefficient of $\xi^{11}$. That this will often be a reasonable assumption may be concluded from the work of Howarth[10], who in considering the flow $u_1 = u_0(1 - \xi)$ found that the coefficients of $\xi^n$ were remarkably similar in shape when n = 5,6,7,8. The equations (2.3) and (2.4) then become

$$\frac{u}{a_1} = \sum_{0}^{4}\xi^{2n+1}\ F'_{2n+1}(\eta) + A(\xi)\ F'_{11}(\eta) \tag{2.5}$$

and

$$T = \sum_{0}^{4}\xi^{2n+1}\ F''_{2n+1}(0) + A(\xi)\ F''_{11}(0) \tag{2.6}$$

With this approximation and the use of a boundary condition at the wall on a differentiated form of the momentum equation the following equation is obtained for T

$$T\,\frac{dT}{d\xi} = P + \frac{F_{11}^{v}(0)}{F_{11}''(0)}\,T \tag{2.7}$$

where
$$P = \sum_{0}^{4} b_{2n+1}\,\xi^{2n+1} \tag{2.8}$$

and
$$b_{2n+1} = F_{2n+1}^{v}(0) - \frac{F_{11}^{v}(0)}{F_{11}''(0)}\,F_{2n+1}''(0) \tag{2.9}$$

Equation (2.7) may be integrated to give

$$T^2 = Q + \frac{2F_{11}^{v}(0)}{F_{11}''(0)}\int_0^{\xi} T\,d\xi \tag{2.10}$$

where
$$Q(\xi) = 2\int_0^{\xi} P\,d\xi = \sum_{0}^{4} \frac{b_{2n+1}}{n+1}\,\xi^{2n+2} \tag{2.11}$$

Equation (2.10) may then be solved by a procedure due to Thwaites[24], (1949), in which $\int_0^{\xi} T\,d\xi$ is replaced by its Simpson's rule equivalent. Hence if $T(\xi)$ and $T(\xi + h)$ are known, the value of $T(\xi + 2h)$ may be determined by solution of a quadratic equation. Having obtained T, the relevant value of $A(\xi)$ is deduced from (2.6) by subtracting off the first five terms and dividing by $F_{11}''(0)$. The non-dimensional displacement thickness is given by

$$\frac{u_1}{(a_1 \nu c)^{\frac{1}{2}}}\,\delta_1 = \lim_{\eta \to \infty}\left\{ \frac{u_1}{a_1}\eta - \sum_{0}^{4} \xi^{2n+1}\, F_{2n+1}(\eta) - A(\xi)\, F_{11}(\eta) \right\} \tag{2.12}$$

The momentum thickness $\delta_2$ can be obtained, using the values of skin friction and displacement thickness, by solving the momentum integral equation.

For the particular flow $u_1 = u_0(\xi + \xi^3)$ we have that $a_1 = u_0$,

$a_3 = u_0$, $a_5 = a_7 = \ldots = 0$. Therefore using the same notation as Curle [3] we have

$$F_1 = f_1;\ F_3 = 4f_3;\ F_5 = 6h_5;\ F_7 = 8k_7;\ F_9 = 10q_9;\ F_{11} = 12n_{11}.$$

This yields the first six terms in the series for the non-dimensional skin-friction as

$$\begin{aligned} T &= f_1''(0)\xi + 4f_3''(0)\xi^3 + 6h_5''(0)\xi^5 + 8k_7''(0)\xi^7 + 10q_9''(0)\xi^9 + 12n_{11}''(0)\xi^{11} + \\ &= 1.232588\xi + 2.89779\xi^3 + 0.71509\xi^5 + 0.06111\xi^7 - 0.3079\xi^9 + 0.6187\xi^{11} + \end{aligned} \tag{2.13}$$

using the values as given by Tifford[25]

The corresponding expansions for $\delta_1$ and $\delta_2$ are

$$\left(\frac{u_0}{\nu c}\right)^{\frac{1}{2}} \delta_1 = \frac{0.6479\xi - 0.113896\xi^3 + 0.44341\xi^5 - 0.79612\xi^7 + 1.3967\xi^9 - 2.49\xi^{11} +}{(\xi + \xi^3)}$$

and

$$\left(\frac{u_0}{\nu c}\right)^{\frac{1}{2}} \delta_2 = \frac{0.292344\xi^2 + 0.266996\xi^4 + 0.102228\xi^6 - 0.059125\xi^8 + 0.068376\xi^{10} - 0.090116\xi^{12} +}{(\xi + \xi^3)^2}$$

The method for solving equation (2.10) which has been indicated above reduces to the formula

$$T(\xi + 2h) = \frac{kh}{6} + \sqrt{\left(T(\xi) + \frac{kh}{6}\right)^2 + \left(Q(\xi + 2h) - Q(\xi)\right) + \frac{4kh}{3}\,T(\xi + h)} \tag{2.14}$$

where h is the step length,

$$k = 2F_{11}^{V}(0)/F_{11}''(0) = -3.328$$

and $Q(\xi) = 3.570193\xi^2 + 9.554395\xi^4 + 10.55662\xi^6 + 4.3204\xi^8 + 0.0041\xi^{10}$

The corresponding series expansion for the displacement thickness is

$$\left(\frac{u_\circ}{\nu_c}\right)^{\frac{1}{2}}\delta_1 = \frac{0.6479\xi-0.113896\xi^3+0.44341\xi^5-0.79612\xi^7+1.3967\xi^9-2.490A(\xi)}{(\xi+\xi^3)}$$

and by solving the momentum integral equation with the above expansions for $\delta_1$ and $\tau_w$, we obtain the following formula for $\delta_2$

$$\left(\frac{u_\circ}{\nu_c}\right)^{\frac{1}{2}}\delta_2 = \frac{(0.292344\xi^2+0.2669965\xi^4+0.102228\xi^6-0.059125\xi^8+0.068376\xi^{10}+1.981773\xi^{12}}{(\xi+\xi^3)^2}$$

$$+\frac{(-0.708783\xi^{14}+0.765236\xi^{16}-0.628512\xi^{18})}{(\xi+\xi^3)^2}$$

$$+A(\xi)\left\{\frac{1.819877A(\xi)-2.068706\xi-2.947302\xi^3+0.773203\xi^5-1.278415\xi^7+2.128755\xi^9}{(\xi+\xi^3)^2}\right\}$$

Results

The non-dimensional skin friction T, is evaluated using the six term series

$T(\xi) = 1.232588\xi+2.89779\xi^3+0.71509\xi^5+0.06111\xi^7-0.3079\xi^9+0.6187\xi^{11}+\ldots$

as far as $\xi = 0.32$.

After this, the non-dimensional skin friction is evaluated using the recurrence procedure as described earlier. Two step sizes are used, 0.01 and 0.02. From the values at common points of the two step-sizes we use Richardson's $h^2$ extrapolation formula to predict 'accurate' values of the skin friction ('extrapolation to zero step size'). From the Richardson's extrapolated value we derive the term $A(\xi)$, which is used in estimating the remaining terms in the series for the velocity u within the boundary layer. This is done by subtracting from the Richardson's extrapolated value the first five terms in the series for skin friction and dividing the result by $F''_{11}(0)$. The first value for which the recurrence technique and the Richardson's $h^2$ method is used is at $\xi = 0.34$. For the step length 0.01 we use the series values at $\xi = 0.31$ and 0.32 to give us the first value by the recurrence relationship at $\xi = 0.33$. Therefore two steps of the recurrence relation are required to get to $\xi = 0.34$. For step length 0.02 we use the series values at $\xi = 0.30$ and $\xi = 0.32$ to give the first value by the recurrence relation at $\xi = 0.34$ i.e. one step by recurrence relation. Richardson's $h^2$ extrapolation formula is then used to give the 'accurate' value. The choice of $\xi = 0.34$ as the first position for the recurrence and Richardson's extrapolated formula technique gives values of $A(\xi)$, which, tabulated to six decimal places increase smoothly, and initially like $\xi^{11}$, which is the first power of $\xi$ which is absorbed into the $A(\xi)$ in the approximate series form for u.

| $\xi$ | $A(\xi)$ | $(\frac{u_0}{\nu c})^{\frac{1}{2}}\delta_1$ | $(\frac{u_0}{\nu c})^{\frac{1}{2}}\delta_2$ | $(\frac{\nu c}{u_0^3})^{\frac{1}{2}}(\frac{\partial u}{\partial y})_w$ |
|---|---|---|---|---|
| 0.00 | 0.000000 | 0.647900 | 0.292344 | 0.000000 |
| 0.04 | 0.000000 | 0.646684 | 0.291837 | 0.049489 |
| 0.08 | 0.000000 | 0.643073 | 0.290329 | 0.100093 |
| 0.12 | 0.000000 | 0.637174 | 0.287860 | 0.152936 |
| 0.16 | 0.000000 | 0.629156 | 0.284493 | 0.209159 |
| 0.20 | 0.000000 | 0.619237 | 0.280311 | 0.269929 |
| 0.24 | 0.000000 | 0.607672 | 0.275412 | 0.336452 |
| 0.28 | 0.000001 | 0.594729 | 0.269898 | 0.409973 |
| 0.32 | 0.000004 | 0.580691 | 0.263882 | 0.491795 |
| 0.36 | 0.000006 | 0.565869 | 0.257511 | 0.583275 |
| 0.40 | 0.000024 | 0.550459 | 0.250839 | 0.685851 |
| 0.44 | 0.000076 | 0.534705 | 0.243977 | 0.801029 |
| 0.48 | 0.000202 | 0.518813 | 0.237016 | 0.930403 |

| $\xi$ | $\lambda$ | l | H | L | μ |
|---|---|---|---|---|---|
| 0.00 | 0.085465 | 0.360339 | 2.21622 | 0.000000 | 0.000000 |
| 0.04 | 0.085578 | 0.360491 | 2.21591 | -0.000592 | 0.000070 |
| 0.08 | 0.085909 | 0.360939 | 2.21498 | -0.002335 | 0.000275 |
| 0.12 | 0.086443 | 0.361660 | 2.21349 | -0.005133 | 0.000602 |
| 0.16 | 0.087152 | 0.362619 | 2.21150 | -0.008845 | 0.001032 |
| 0.20 | 0.088003 | 0.363769 | 2.20911 | -0.013290 | 0.001541 |
| 0.24 | 0.088959 | 0.365068 | 2.20641 | -0.018261 | 0.002103 |
| 0.28 | 0.089978 | 0.366452 | 2.20353 | -0.023548 | 0.002692 |
| 0.32 | 0.091025 | 0.367879 | 2.20057 | -0.028958 | 0.003284 |
| 0.36 | 0.092094 | 0.369353 | 2.19756 | -0.034414 | 0.003862 |
| 0.40 | 0.093122 | 0.370772 | 2.19447 | -0.039650 | 0.004409 |
| 0.44 | 0.094097 | 0.372122 | 2.19162 | -0.044591 | 0.004913 |
| 0.48 | 0.095006 | 0.373389 | 2.18894 | -0.049169 | 0.005368 |

## Section III: Asymptotic Theory for $u_1 = u_o(\xi + \xi^3)$

This section deals with the analysis for large values of $\xi$. A solution is developed in terms of a co-ordinate perturbation expansion from the Falkner-Skan flow $u_1 = u_o\xi^3$. By means of the work of Libby and Chen the form of this expansion is investigated and the solutions of the equations governing the coefficients of the first and second terms in this expansion are given. The asymptotic forms of the parameters are investigated and it is shown how critically these depend on the value of the coefficient of the second term in the asymptotic series for skin friction.

For the flow under consideration the external velocity distribution is $u_1 = u_o(\xi + \xi^3)$. For large values of $\xi$ the flow within the boundary layer may be expected to behave like that under the influence of an external velocity distribution $u_1 = u_o\xi^3$. This is one of the Falkner Skan family of solutions where $u_1 = u_o\xi^n$. The behaviour of these solutions is well-known and it may therefore be useful to express the velocity u within the boundary layer with external flow $u_1 = u_o(\xi + \xi^3)$, in terms of the Falkner-Skan solution plus a small correction term, i.e. $u = u_o\xi^3 f'(\eta) + u'$ where the function f satisfies the equation

$$f''' + ff'' + \frac{3}{2}(1 - f'^2) = 0,\ f(0) = f'(0) = 0,\ f' \to 1 \text{ as } \eta \to \infty$$

$$\text{and } u' << u_o\xi^3 f'(\eta), \text{ and } \eta = \left(\frac{2u_o}{\nu c}\right)^{\frac{1}{2}}\xi y$$

The most convenient form for u' would be $u' = u_o\xi g'(\eta)$ plus small terms where $g'(0) = g(0) = 0$ and $g' \to 1$ as $\eta \to \infty$. This is a possible form for u' but to ensure that no higher order terms have been omitted the following analysis must be carried out.

Define the stream function $\psi$ for the flow with external velocity $u_1 = u_o(\xi + \xi^3)$ to be of the form

$$\psi = \left(\frac{u_o\nu c}{2}\right)^{\frac{1}{2}}\xi^2 f(\eta) + G(\eta, \xi)$$

where $f(\eta)$ satisfies the equation

$$f''' + ff'' + \frac{3}{2}(1 - f'^2) = 0,\ f(0) = f'(0) = 0, f' \to 1 \text{ as } \eta \to \infty$$

This yields the following forms for the velocity components, and their derivatives, within the boundary layer.

$$u = u_o\xi^3 f'(\eta) + \left(\frac{2u_o}{\nu c}\right)^{\frac{1}{2}}\xi G_\eta$$

$$v = -\left(\frac{u_o\nu}{2c}\right)^{\frac{1}{2}}\xi\left\{2f + \eta f'\right\} - \frac{1}{c}\left\{G_\eta \cdot \eta/\xi + G_\xi\right\}$$

$$\frac{\partial u}{\partial x} = u_o\xi^2 f''(\eta) \cdot \eta/c + \frac{3u_o}{c}\xi^2 f' + \frac{1}{c}\left(\frac{2u_o}{\nu c}\right)^{\frac{1}{2}}\left\{G_\eta + \eta G_{\eta\eta} + \xi G_{\xi\eta}\right\}$$

$$\frac{\partial u}{\partial y} = \left(\frac{2u_\circ}{\nu c}\right)^{\frac{1}{2}} u_\circ \xi^4 f'' + \left(\frac{2u_\circ}{\nu c}\right)\xi^2 \, G_{\eta\eta}$$

$$\frac{\partial^2 u}{\partial y^2} = \left(\frac{2u_\circ}{\nu c}\right) u_\circ \xi^5 f''' + \left(\frac{2u_\circ}{\nu c}\right)^{3/2} \xi^3 \, G_{\eta\eta\eta}$$

On substituting into the boundary layer equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_1 \frac{du_1}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

and collecting together the coefficients of powers of $\xi$ and setting $G = \left(\frac{\nu u_\circ c}{2}\right)^{\frac{1}{2}} H$ the following is obtained where $H \to 1$ as $\eta \to \infty$

$H_\xi(\xi,0) = H_\eta(\xi,0) = 0$

Coefficient of $\xi^5$:- $\frac{2u_\circ^2}{c}\left\{ f''' + ff'' + \frac{3}{2}(1 - f'^2)\right\} = 0$ by hypothesis.

Coefficient of $\xi^4$: $\frac{u_\circ^2}{c}\left\{ f'H_{\xi\eta} - f''H_\xi \right\}$

Coefficient of $\xi^3$: $\frac{u_\circ^2}{c}\left\{ 4f'H_\eta - 2fH_{\eta\eta} - 2H_{\eta\eta\eta} - 4 \right\}$

Coefficient of $\xi^2$: $\frac{u_\circ^2}{c}\left\{ H_\eta H_{\xi\eta} - H_{\eta\eta} H_\xi \right\}$

Coefficient of $\xi$ : $\frac{u_\circ^2}{c}\left\{ H_\eta^2 - 1 \right\}$

$\therefore$ The total expression reduces to

$$(H_\eta^2 - 1)\xi + (H_\eta H_{\xi\eta} - H_{\eta\eta}H_\xi)\xi^2 + (4f'H_\eta - 2H_{\eta\eta}f - 2H_{\eta\eta\eta} - 4)\xi^3 + (f'H_{\xi\eta} - f''H_\xi)\xi^4 = 0 \qquad ***$$

subject to $H_\eta(\xi,0) = 0$, $H_\xi(\xi,0) = 0$

$H_\eta \to 1$ as $\eta \to \infty$

Suppose we now try to find what forms for H are admissible for this equation. (A) First of all, if we look for u of the form

$$u = u_\circ\xi^3 f' + u_\circ\xi^{\alpha+1} \log\xi^2 h' + u_\circ\xi g' + \ldots$$

where $\alpha < 2$

this means taking $H = \xi^{\alpha} \log\xi^2 + g$. On substituting this expression for H into equation (***) we find that the coefficients of $\xi$ yield the following forms

Coefficient of $\xi^{\alpha+3}\log\xi^2$:- $4f'h' - 2fh'' - 2h''' + \alpha\,(f'h' - f''h)$

Coefficient of $\xi^{\alpha+3}$:- $2(f'h' - f''h)$

Coefficient of $\xi^{3}$:- $4f'g' - 2fg'' - 2g''' - 4$

Coefficient of $\xi^{2\alpha+1}(\log\xi^2)^2$:- $h'^2 + \alpha(h'g' - hh'')$

Coefficient of $\xi^{2\alpha+1}\log\xi^2$:- $2\left\{h'^2 - hh''\right\}$

Coefficient of $\xi^{\alpha+1}\log\xi^2$:- $2h'g' + \alpha(h'g' - g''h)$

$\xi^{\alpha+1}$:- $2(h'g' - hg'')$

$\xi$ :- $(g')^2 - 1$

If we now consider the equation which results when we set the coefficient of $\xi^{\alpha+3}\log\xi^2$ equal to zero, we obtain

$$4f'h' - 2fh'' - 2h''' + \alpha(f'h' - f''h) = 0 \qquad (**)$$

$$\text{with } h(0) = h'(0) = 0,\ h' \to 0 \text{ as } \eta \to \infty$$

This has a solution $h = f'$ but this is not an eigen-solution since $f''(0) \neq 0$. Now if we define $\alpha = 2(1 - \lambda)$ we get

$$4f'h' - 2fh'' - 2h''' + 2(1 - \lambda)(f'h' - f''h) = 0$$

$$\therefore\ 2f'h' - fh'' - h''' + (1 - \lambda)(f'h' - f''h) = 0$$

$$\therefore\ h''' + fh'' + (\lambda - 3)f'h' + (1 - \lambda)f''h = 0 \qquad (*)$$

But equation ( * ) is just equation (27) given by Libby and Chen on p. 275 of J.F.M., Vol. 33, Part 2, for the case $\beta = \frac{3}{2}$. Also for the equation on p. 279 of the same journal a table is given of the lowest eigenvalue ($\lambda_1$) for several values of $\beta$. The values of $\lambda_1$ increase monotonically with $\beta$ giving $\lambda_1 = 4.177$, and 6.131 for $\beta = 1$ and 2, respectively.

Therefore for the case $\beta = \frac{3}{2}$ all the eigenvalues will be positive with the lowest one lying between 4.177 and 6.131. Now since $\alpha = 2(1 - \lambda)$, the largest value of $\alpha$ will lie between $\alpha = -10.262$ and $\alpha = -6.354$. This suggests that in the expansion for the stream function in the boundary layer we do not expect terms with dependence on $\xi$ of the form $\xi^{\alpha} \log \xi^{2}$ to appear until we have negative values of the exponent $\alpha$, with $\alpha$ no larger than -6.

(B) Next we look for u of the form

$$u = u_0 \xi^3 f' + u_0 \xi^{\alpha+1} h' + u_0 \xi g' + \ldots$$

$$\text{for } \alpha < 2$$

To obtain this we take $H = \xi^{\alpha} h + g$ and obtain the following as coefficients of powers of $\xi$.

Coefficient of $\xi^{\alpha+3}$:- $4f'h' - 2fh'' - 2h''' + \alpha(h'f' - f''h)$

Coefficient of $\xi^{2\alpha+1}$:- $(h')^2 + \alpha\,(h'h - hh'')$

Coefficient of $\xi^{\alpha+1}$:- $2h'g' + \alpha\,(h'g' - g''h)$

$\xi^3$:- $4f'g' - 2fg'' - 2g'' - 2g''' - 4$

$\xi$:- $g'^2 - 1$

On setting the coefficient of $\xi^{\alpha+3}$ equal to zero we obtain equation From the analysis of that equation it is seen that $\alpha$ may be expected to be negative. This suggests that in the expression for the velocity u within the boundary layer in the form.

$$u = u_0\xi^3 f'(\eta) + u_0\xi^{\alpha+1}h'(\eta) + u_0\xi g'(\eta) + \ldots$$

there will be a non-trivial solution for $h'(\eta)$ only when $\alpha$ is negative.

(C) If we now combine (A) and (B) we try to find an expansion for u of the form

$$u = u_0\xi^3 f'(\eta) + u_0\xi^{\alpha+1}\log\xi^2 h' + u_0\xi^{\gamma+1}p' + \ldots$$

say where $\alpha < 2$, $\gamma < 2$.

To obtain this we take $H = \xi^{\alpha}\log\xi^2 h + \xi^{\gamma}p$ and obtain the following as coefficients of powers of $\xi$.

Coefficients of $\xi^{2\alpha+1}(\log\xi^2)^2$:- $h'^2 + \left\{\alpha(h')^2 - \alpha hh''\right\}$

$\xi^{\alpha+\gamma+1}(\log\xi^2)$:- $2h'p' + \left\{(\gamma+\alpha)p'h' - \gamma ph'' - \alpha p''h\right\}$

$\xi^{2\alpha+1}\log\xi^2$:- $2(h'^2 - hh'')$

$\xi^{\alpha+\gamma+1}$:- $2h'p' - 2p''h$

$\xi^{\alpha+3}\log\xi^2$:- $4f'h' - 2fh'' - 2h''' + \alpha f'h' - \alpha f''h$

$\xi^{\gamma+3}$:- $4f'p' - 2fp'' - 2p''' + \gamma f'p' - \gamma pf''$

$\xi^{2\gamma+1}$:- $\gamma(p'^2 - pp'') + p'^2$

$\xi^{\alpha+3}$:- $2f'h' - f''h$

$\xi^3$:- 4; $\xi$:- 1

The setting of the coefficients of $\xi^{\alpha+3}\log\xi^2$ and $\xi^{\gamma+3}$ equal to zero yields equation ( $*$ ) for both $\alpha$ and $\gamma$ in terms of f and h, and in terms of f and p respectively. As has been seen previously this suggests that $\alpha$ and $\gamma$ should both be negative.

From this analysis it seems reasonable to choose the stream function within the boundary layer, for external flow $u_1 = u_\circ(\xi + \xi^3)$, to be of the form

$$\psi = \psi_F + \psi' = \left(\frac{u_\circ \nu c}{2}\right)^{\frac{1}{2}} \xi^2 f(\eta) + \left(\frac{u_\circ \nu c}{2}\right)^{\frac{1}{2}} g(\eta) + \ldots$$

where $f(\eta)$ satisfies the equation

$$f''' + ff'' + \frac{3}{2}(1 - f'^2) = 0,\ f(0) = f'(0) = 0,\ f' \to 1 \text{ as } \eta \to \infty$$

and $g(\eta)$ is such that $f(0) = g'(0) = 0$ and $g' \to 1$ as $\eta \to \infty$.

III.1 <u>The functions $f(\eta)$, $g(\eta)$</u>

From the preceding analysis it was decided that the form for the stream function $\psi(\xi,\eta)$, within the boundary layer should be (where $\eta = \left(\frac{2u_\circ}{\nu c}\right)^{\frac{1}{2}} \xi y$)

$$\begin{aligned}\psi(\xi,\eta) &= \psi_F(\xi,\eta) + \psi'(\xi,\eta) \\ &= \left(\frac{u_\circ \nu c}{2}\right)^{\frac{1}{2}} \xi^2 f(\eta) + \left(\frac{u_\circ \nu c}{2}\right)^{\frac{1}{2}} g(\eta) + \end{aligned} \qquad (1)$$

where $f(\eta)$ satisfies the equation

$$f''' + ff'' + \frac{3}{2}(1 - f'^2) = 0;\ f(0) = f'(0) = 0;\ f' \to 1 \text{ as } \eta \to \infty \quad (2)$$

and by substituting (1) into the boundary layer equations and equating coefficients of powers of $\xi$ to zero, the function $g(\eta)$ satisfies the equation

$$g''' - 2f'g' + fg'' = -2;\ g(0) = g'(0) = 0;\ g' \to 1 \text{ as } \eta \to \infty \quad (3)$$

A solution to equation (2) was produced for me by Miss S. M. Picken N.P.L. by the method of selected points which she describes in reference 14. The values obtained are given tabulated on page 50.

It is interesting to note (reference 16) that for large values of $\eta$ the function $f(\eta)$ and its derivatives behave as

$$f(\eta) \sim (\eta - c) + B\,\frac{e^{-\frac{1}{2}(\eta - c)^2}}{(\eta - c)^5} + \ldots \quad , \text{ where c is a constant.}$$

$$f'(\eta) \sim 1 - B\,\frac{e^{-\frac{1}{2}(\eta - c)^2}}{(\eta - c)^4} + \ldots$$

$$f''(\eta) \sim B\,\frac{e^{-\frac{1}{2}(\eta - c)^2}}{(\eta - c)^3} + \ldots ; \quad f'''(\eta) \sim -B\,\frac{e^{-\frac{1}{2}(\eta - c)^2}}{(\eta - c)^2} + \ldots$$

where B and C are constants.

The equation $g''' - 2f'g' + fg'' = -2$, $g(0) = g'(0) = 0$, $g' \to 1$ as $\eta \to \infty$ was solved as follows. Set $p = g'$ so that the equation becomes

$$p'' - 2f'p + fp' = -2 \text{ where } p(0) = 0;\ p(\infty) = 1$$

Now use the central difference formulae, using a step length of h, for $p'$ and $p''$, which are

$$p'' = \frac{p_{j+1} - 2p_j + p_{j-1}}{h^2}\ ; \quad p' = \frac{p_{j+1} - p_{j-1}}{2h} \quad 2 \leqslant j \leqslant N$$

where $\qquad p_j = p(x_j),\ f_j = f(x_j);\ \alpha_j = f'(x_j).$

Using this discretisation the equation becomes

$$\frac{p_{j+1} - 2p_j + p_{j-1}}{h^2} - 2\alpha_j p_j + f_j\left\{\frac{p_{j+1} - p_{j-1}}{2h}\right\} = -2$$

$$\therefore\ p_{j+1} - 2p_j + p_{j-1} - 2h^2\alpha_j p_j + \tfrac{1}{2}hf_j p_{j+1} - \frac{h}{2}f_j p_{j-1} = -2h^2$$

$$\therefore\ p_{j-1}\left\{1 - \frac{h}{2}f_j\right\} - 2(1 + h^2\alpha_j)p_j + (1 + \tfrac{1}{2}hf_j)p_{j+1} = -2h^2$$

This yields the system of equations(with $p_1 = 0$, $p_{N+1} = 1$) $j = 2, N$

$$\begin{pmatrix} -2(1 + h^2\alpha_2) & (1 + \frac{1}{2}hf_2) & \cdots\cdots & & \\ (1 - \frac{h}{2}f_3) & -2(1 + h^2\alpha_3) & (1 + \frac{1}{2}hf_3) & \cdots & \\ & & & \vdots & \\ & & & (1 - \frac{1}{2}hf_N) & -2(1 + h^2\alpha_N) \end{pmatrix} \begin{pmatrix} p_2 \\ \\ \vdots \\ \\ p_N \end{pmatrix} = \begin{pmatrix} -2h^2 \\ \\ \vdots \\ \\ -2h^2 - (1 + \frac{1}{2}hf_j) \end{pmatrix}$$

The results for the equation $g''' - 2f'g' + fg'' = -2$; $g(0) = g'(0) = 0$, $g' \to 1$ as $\eta \to \infty$ where f satisfies the equation $f''' + ff'' + \frac{3}{2}(1 - f'^2) = 0$: $f(0) = f'(0) = 0$, $f' \to 1$ as $\eta \to \infty$ty are given on the following page.

| $\eta$ | $g'(\eta)$ | $g(\eta)$ | $\eta$ | $g'(\eta)$ | $g(\eta)$ |
|---|---|---|---|---|---|
| 0.0 | 0.000000 | 0.000000 | 0.1 | 0.161246 | 0.008230 |
| 0.2 | 0.302835 | 0.031594 | 0.3 | 0.425670 | 0.068172 |
| 0.4 | 0.531032 | 0.116146 | 0.5 | 0.620436 | 0.173846 |
| 0.6 | 0.695519 | 0.239755 | 0.7 | 0.757942 | 0.312528 |
| 0.8 | 0.809331 | 0.390976 | 0.9 | 0.851223 | 0.474077 |
| 1.0 | 0.885041 | 0.560951 | | | |
| 1.2 | 0.933462 | 0.743178 | | | |
| 1.4 | 0.963196 | 0.933095 | | | |
| 1.6 | 0.980681 | 1.127643 | | | |
| 1.8 | 0.990483 | 1.324858 | | | |
| 2.0 | 0.995686 | 1.523533 | | | |
| 2.2 | 0.998272 | 1.722960 | | | |
| 2.4 | 0.999453 | 1.922749 | | | |
| 2.6 | 0.999928 | 2.122695 | | | |
| 2.8 | 1.000079 | 2.322699 | | | |
| 3.0 | 1.000099 | 2.522718 | | | |
| 3.2 | 1.000078 | 2.722736 | | | |
| 3.4 | 1.000051 | 2.922749 | | | |
| 3.6 | 1.000029 | 3.122757 | | | |
| 3.8 | 1.000016 | 3.322761 | | | |
| 4.0 | 1.000008 | 3.522763 | | | |
| 4.2 | 1.000004 | 3.722764 | | | |
| 4.4 | 1.000002 | 3.922765 | | | |
| 4.6 | 1.000001 | 4.122765 | | | |
| 4.8 | 1.000000 | 4.322765 | | | |
| 5.0 | 1.000000 | 4.522765 | | | |
| 5.2 | 1.000000 | 4.722765 | | | |
| 5.4 | 1.000000 | 4.922765 | | | |
| 5.6 | 1.000000 | 5.122765 | | | |
| 5.8 | 1.000000 | 5.322765 | | | |
| 6.0 | 1.000000 | 5.522765 | | | |

$$\mathop{\mathrm{lt}}_{\eta\to\infty}(\eta - g(\eta)) = 0.47723$$

For the purposes of this problem $\infty$ was taken to be where the independent variable attained the value 6.0 and the step length was chosen to be h = 0.1 as the values of f and f' had been given tabulated by Miss Picken at this interval. A table of the solution is given above.

Solution of the equation

$$f''' + ff'' + \frac{3}{2}(1 - f'^2) = 0; \; f(0) = f'(0) = 0, \; f' \to 1 \text{ as } \eta \to \infty$$

| $\eta$ | f | f' | f" | f''' |
|---|---|---|---|---|
| 0.0 | 0.000000 | 0.000000 | 1.477224 | -1.5000 |
| 0.1 | 0.007136 | 0.140240 | 1.327906 | -1.479976 |
| 0.2 | 0.027555 | 0.265707 | 1.182234 | -1.426679 |
| 0.3 | 0.059803 | 0.376928 | 1.043372 | -1.349284 |
| 0.4 | 0.102491 | 0.474670 | 0.913021 | -1.255610 |
| 0.5 | 0.154318 | 0.559863 | 0.792575 | -1.152139 |
| 0.6 | 0.214080 | 0.633539 | 0.682744 | -1.044104 |
| 0.7 | 0.280678 | 0.696774 | 0.583769 | -0.935610 |
| 0.8 | 0.353123 | 0.750651 | 0.495533 | -0.829768 |
| 0.9 | 0.430532 | 0.796226 | 0.417650 | -0.728848 |
| 1.0 | 0.512125 | 0.834507 | 0.349546 | -0.634408 |
| 1.2 | 0.685230 | 0.892888 | 0.239793 | -0.468441 |
| 1.4 | 0.868027 | 0.932422 | 0.160007 | -0.334773 |
| 1.6 | 1.057302 | 0.958463 | 0.103832 | -0.231804 |
| 1.8 | 1.250790 | 0.975143 | 0.065501 | -0.155572 |
| 2.0 | 1.446941 | 0.985527 | 0.040147 | -0.101196 |
| 2.2 | 1.644729 | 0.991806 | 0.023893 | -0.063779 |
| 2.4 | 1.843492 | 0.995493 | 0.013799 | -0.038929 |
| 2.6 | 2.042821 | 0.997593 | 0.007728 | -0.023001 |
| 2.8 | 2.242467 | 0.998752 | 0.004195 | -0.013148 |
| 3.0 | 2.442286 | 0.999373 | 0.002057 | -0.007268 |
| 3.2 | 2.642197 | 0.999694 | 0.001123 | -0.003883 |
| 3.4 | 2.842153 | 0.999856 | 0.000553 | -0.002004 |
| 3.6 | 3.042133 | 0.999934 | 0.000263 | -0.000999 |
| 3.8 | 3.242124 | 0.999971 | 0.000121 | -0.000481 |
| 4.0 | 3.442120 | 0.999987 | 0.000054 | -0.000223 |
| 4.2 | 3.642119 | 0.999995 | 0.000023 | -0.000100 |
| 4.4 | 3.842118 | 0.999998 | 0.000010 | -0.000043 |
| 4.6 | 4.042118 | 0.999999 | 0.000004 | -0.000018 |
| 4.8 | 4.242118 | 1.000000 | 0.000001 | -0.000007 |
| 5.0 | 4.442118 | 1.000000 | 0.000000 | -0.000003 |
| 5.2 | 4.642118 | 1.000000 | 0.000000 | -0.000001 |
| 5.4 | 4.842118 | 1.000000 | 0.000000 | -0.00000037 |
| 5.6 | 5.042118 | 1.000000 | 0.000000 | -0.000000 |
| 5.8 | 5.242118 | 1.000000 | 0.000000 | -0.000000 |
| 6.0 | 5.442118 | 1.000000 | 0.000000 | -0.000000 |

$$\operatorname{lt}_{\eta \to \infty}(\eta - f) = 0.557882$$

$$f''(0) = 1.477224$$

The $\underset{\eta\to\infty}{\mathrm{lt}}(\eta - g(\eta))$ was easily obtained by integration and the value of 0.4772(3) may be expected to be correct to at least four figures.

The value of $g''(0)$, which is required for the calculation of skin friction, was obtained in the following two ways. In the equation

$$g''' - 2f'g' + fg'' = -2$$

write $g' = f''h$. This gives the equation for h

$$h'' + \left(2\frac{f'''}{f''} + f\right)h' = -\frac{2}{f''}$$

which has an integrating factor $(f'')^2 P$ where $P(\eta) = e^{\int_0^\eta f(\alpha)d\alpha}$

$$\therefore h'(f'')^2 P = A - 2\int_0^\eta Pf''d\eta$$

When $\eta = 0$, this yields $h'(0) = \dfrac{A}{(f''(0))^2}$

The value of A may be determined as follows.

$$\text{As } \eta \to \infty,\ g' \to 1 \quad \therefore hf'' \to 1$$

$$\therefore h \sim \frac{1}{f''} \text{ for large } \eta$$

$$\therefore h' \sim -\frac{f'''}{(f'')^2}$$

$$\therefore h'(f'')^2 \to -f''' \text{ as } \eta \to \infty$$

But from an analysis of the equation governing $f(\eta)$ it is known that for large

$$f \sim (\eta - c) + B\frac{e^{-\frac{1}{2}(\eta - c)^2}}{(\eta - c)^5}$$

$$\text{and } f'' \sim B\,\frac{e^{-\frac{1}{2}(\eta - c)^2}}{(\eta - c)^3}$$

$$\therefore\ P = e^{\int f d\eta} \sim e^{\frac{1}{2}(\eta - c)^2} \text{ for large } \eta$$

$$\therefore \text{ As } \eta \to \infty,\ h'(f'')^2 P \to -f'''P$$

$$\to B\,\frac{e^{-\frac{1}{2}(\eta - c)^2} \cdot e^{\frac{1}{2}(\eta - c)^2}}{(\eta - c)^2}$$

$$\to 0 \text{ as } \eta \to \infty$$

This therefore gives $A = 2\int_0^\infty f''(\xi)\ e^{\int_0^\xi f(\alpha)d\alpha} d\xi$

$$\text{Now } g' = f''h$$

$$\therefore\ g'' = f''h' + hf'''$$

$$g''(0) = f''(0)h'(0) + h(0)f'''(0)$$

But $h(0) = 0$ since $g(0) = 0 = f''(0).h(0)$ and $f''(0) \neq 0$

$$\therefore\ g''(0) = f''(0)h'(0)$$

$$= \frac{2\int_0^\infty f''(\xi)\ e^{\int_0^\xi f(\alpha)d\alpha} d\xi}{f''(0)}$$

This may be computed using Miss Picken's results and it is found that by this method $g''(0) = 1.6944$. Alternatively the following procedure may be adopted.

The equation is given as $g''' - 2f'g' + fg'' = -2$ (1)

with boundary conditions $\underline{g(0) = g'(0) = 0}$; $g' \to 1$ as $\eta \to \infty$

Setting $\eta = 0$ in (1) yields $\underline{g'''(0) = -2}$ (1A)

Now differentiate (1) wrt $\eta$, to get

$$g^{1V} - f'g'' - 2g'f'' + fg''' = 0 \qquad (2)$$

Set $\eta = 0$ to get $\underline{g^{1V}(0) = 0}$ (2A)

Differentiate (2) wrt $\eta$, to get

$$g^{(V)} - 3f''g'' - 2g'f''' + fg^{(1V)} = 0 \qquad (3)$$

Set $\eta = 0$, and using $f''(0) = 1.477224$, $f'''(0) = -\frac{3}{2}$ obtain

$$g^{(V)}(0) - 3f''(0)g''(0) = 0 \therefore g^{(V)}(0) = 3f''(0).g''(0) = 3 \times 1.477224g''(0)$$

$$= 4.431672g''(0) \qquad (3A)$$

Differentiate (3) wrt $\eta$ to get

$$g^{(V1)} - 5f'''g'' - 3f''g''' - 2g'f^{1V} + fg^{V} + g^{(1V)}f' = 0 \qquad (4)$$

Set $\eta = 0$ and using the above results it is found that

$$g^{V1}(0) + 5 \times \frac{3}{2} g''(0) + 6f''(0) = 0$$

which gives $g^{(V1)}(0) = -\frac{7}{2} g''(0) - 6 \times 1.477224$

$$= \underline{-3.5g''(0) - 8.863344} \qquad (4A)$$

Differentiate (4) wrt $\eta$ to get

$$g^{(7)} - 8f'''g''' - 7g''f^{1V} - 2g^{1V}f'' - 2g'f^{(V)} + g^{(V)}f' = 0 \qquad (5)$$

which on setting $\eta = 0$, gives

$$\underline{g^{(7)}(0) = 8f'''(0)g'''(0) = 24} \qquad (5A)$$

Now for small values of $\eta$, by applying Taylor's theorem we have

$$g'(\eta) = g'(0) + \eta g''(0) + \frac{\eta^2}{2!} g'''(0) + \frac{\eta^3}{3!} g^{1V}(0) + \frac{\eta^4}{4!} g^{(V)}(0) + \frac{\eta^5}{5!} g^{(V1)}(0) + \frac{\eta^6}{6!} g^{(V11)}(0) + \ldots$$

$\therefore$ Applying the results of (1A), (2A), etc., we obtain the following series.

$$g'(\eta) = \eta g''(0) - \eta^2 + 0.184653\eta^4 g''(0) + \left(-0.029167g''(0)\eta^5 - 0.0738612\eta^5\right) + 0.033333\eta^6 + \ldots$$

But from computed results we have $g'(\eta)$ and for $\eta = 0.1$, $g'(0.1) = 0.1612456$.

Using 1st term

$$g'(0.1) = 0.1612456 = 0.1g''(0) \text{ gives } \underline{g''(0) = 1.612456}$$

Using 1st 2 terms

$$g'(0.1) = 0.1g''(0) - 0.01 \text{ gives } \underline{g''(0) = 1.712456}$$

Using 1st 3 terms

$$g'(0.1) = 0.1612456 = 0.1g''(0) - 0.01 + 0.0000184653g''(0)$$

$$\text{which gives } \underline{g''(0) = 1.712139}$$

Using 1st 4 terms

$$g'(0.1) = 0.1612456 = 0.1g''(0) - 0.01 + 0.0000184653g''(0) - 0.00000029g''(0) - 0.00000074$$

$$\text{which gives } \underline{g''(0) = 1.712142}$$

Using 1st 5 terms

$$g'(0.1) = 0.1612456 = 0.1g''(0) - 0.01 + 0.0000184653g''(0) - 0.00000029g''(0) - 0.00000074 + 0.00000003$$

$$\text{which gives } \underline{g''(0) = 1.712142}$$

As there was a slight discrepancy between the values of $g''(0)$ as given by the two methods, it was thought to be useful to examine the effect which a slight error in $g''(0)$ might have on the parameters. This was done as follows.

III.2 <u>The Generalised Asymptotic Expansions</u>

With u of the form $u = u_0\xi^3 f'(\eta) + u_0\xi g'(\eta) + ---$ for large values of $\xi$, the asymptotic form of the skin friction is given by

$$T = \left(\frac{\nu c}{u_o^3}\right)^{\frac{1}{2}}\left(\frac{\partial u}{\partial y}\right)_w = \sqrt{2}\, f''(0)\xi^4 + \sqrt{2}\, g''(0)\xi^2 + \dots$$

and the corresponding form for the non-dimensional displacement thickness is

$$I_1 = \left(\frac{u_o}{\nu c}\right)^{\frac{1}{2}} \delta_1 = \frac{A + B\xi^2 + \dots}{\xi + \xi^3}$$

$$\text{where } A = \frac{1}{\sqrt{2}} \operatorname*{lt}_{\eta\to\infty}(\eta - g(\eta));\ B = \frac{1}{\sqrt{2}} \operatorname*{lt}_{\eta\to\infty}(\eta - f(\eta))$$

and substituting in the known numerical values

$$f''(0) = 1.477224,\ \operatorname*{lt}_{\eta\to\infty}(\eta - f) = 0.557882;\ \operatorname*{lt}_{\eta\to\infty}(\eta - g) = 0.47723$$

and leaving $\alpha_1 = \sqrt{2}\, g''(0)$ we obtain

$$T = 2.089090\xi^4 + \alpha_1\xi^2 + \dots$$

$$I_1 = (0.394482\xi^2 + 0.337452 + \dots)/(\xi + \xi^3)$$

The momentum integral equation becomes, $\left(I_2 = \left(\frac{u_o}{\nu c}\right)^{\frac{1}{2}} \delta_2\right)$

$$\frac{d}{d\xi}\left[(\xi + \xi^3)^2\, I_2\right] = T(\xi) - I_1(\xi)(\xi + \xi^3)(1 + 3\xi^2)$$

On performing the necessary algebra this yields

$$I_2(\xi) = \frac{0.18113}{\xi} + \left(\frac{\alpha_1 - 2.49364}{3}\right)\frac{1}{\xi^3}$$

In terms of these quantities $I_1$, $I_2$ $T$ the required forms are

$$\lambda(\xi) = (1 + 3\xi^2)I_2^2;\ l(\xi) = T(\xi)I_2(\xi)/\xi + \xi^3;\ H(\xi) = I_1/I_2$$

$$\text{Now} \quad I_2^2 = \frac{0.03281}{\xi^2} + 0.120755(\alpha_1 - 2.493636)\frac{1}{\xi^4} + \dots$$

$$\therefore \lambda(\xi) = (1 + 3\xi^2)\left(\frac{0.032809}{\xi^2} + 0.120755(\alpha_1 - 2.49363)\frac{1}{\xi^4} + \dots\right.$$

$$= 0.098427 + (0.362265\alpha_1 - 0.870548)\frac{1}{\xi^2} + \dots$$

Similarly $l = 0.378403 + (0.877496\alpha_1 - 2.114879)\frac{1}{\xi^2} + \dots$

and $$H = \frac{0.394482\xi^{-1} - 0.057030\xi^{-3} + \dots}{0.18113\xi^{-1} + \frac{(\alpha_1 - 2.493636)}{3}\xi^{-3} + \dots}$$

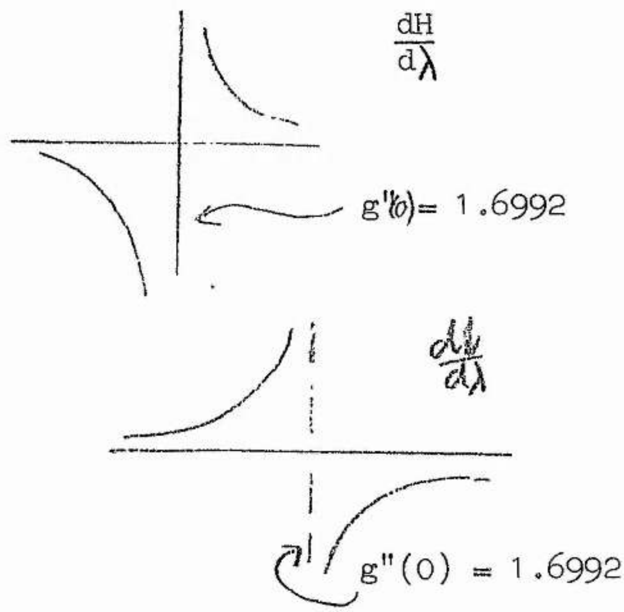
$$= 2.177858 + (9.679249 - 4.007843\alpha_1)\xi^{-2} + \dots$$

From the above expressions, using $\frac{dH}{d\lambda} = \frac{dH}{d\xi^2} \cdot \frac{d\xi^2}{d\lambda}$ and $\frac{dl}{d\lambda} = \frac{dl}{d\xi^2} \cdot \frac{d\xi^2}{d\lambda}$ it is found that

$$\frac{dH}{d\lambda} = \frac{4.007843\alpha_1 - 9.679249}{0.870548 - 0.362265\alpha_1} \; ; \; \frac{dl}{d\lambda} = \frac{2.114879 - 0.877496\alpha_1}{0.870548 - 0.362265\alpha_1}$$

These give the following table.

| $g''(0)$ | $\frac{dH}{d\lambda}$ | $\frac{dl}{d\lambda}$ |
|---|---|---|
| 1.600 | -12.00982 | 2.54411 |
| 1.610 | -12.11587 | 2.55775 |
| 1.620 | -12.24870 | 2.57486 |
| 1.630 | -12.41987 | 2.59688 |
| 1.640 | -12.64881 | 2.62637 |
| 1.650 | -12.97077 | 2.66780 |
| 1.660 | -13.45672 | 2.73034 |
| 1.670 | -14.2747 | 2.83569 |
| 1.680 | -15.94314 | 3.0504 |
| 1.690 | -21.21796 | 3.729503 |
| 1.700 | 114.44474 | -13.734 |




From the above table it is seen that $\frac{dH}{d\lambda}$ and $\frac{dl}{d\lambda}$ become infinite when $\alpha_1 = 2.40307$ which corresponds to $g''(0) = 1.6992$. From the calculations it appears that $g''(0) \in [1.69, 1.712]$ and therefore these gradients which are critically dependent on the value of $g''(0)$ may have large errors.

## Section IV: Calculation of the quantities H, l, L as functions of $\lambda$

This section deals with the approach to the problem which was finally adopted to give the required parameters for the flow $u_1 = u_0$

$(\xi+\xi^3)$ and is as follows. In the range of our interest in the parameters it was observed from the work of others that H, l and L were almost linear functions of $\lambda$. What was therefore done was to express $\lambda$, H and l as functions of $\xi$ which was then eliminated to give $H(\lambda)$, $l(\lambda)$ and using $L = 2(1-\lambda(H+2))$, $L(\lambda)$. The series expansions which resulted gave good agreement with the known limiting values and only minor corrections were necessary to make the agreement complete.

If we denote by T, $I_1$ and $I_2$ the non-dimensional skin friction $\left(\left(\frac{\nu c}{u_o^3}\right)^{\frac{1}{2}}\left(\frac{\partial u}{\partial y}\right)_w\right)$, the non-dimensional displacement thickness $\left(\left(\frac{u_o}{\nu c}\right)^{\frac{1}{2}}\delta_1\right)$ and the non-dimensional momentum thickness $\left(\left(\frac{u_o}{\nu c}\right)^{\frac{1}{2}}\delta_2\right)$ respectively the series expansions for these quantities in terms of $\xi$ are, for $u_1 = u_o(\xi+\xi^3)$

$$T(\xi) = 1.232588\xi + 2.89779\xi^3 + 0.71509\xi^5 + 0.06111\xi^7 - 0.3079\xi^9 + 0.6187\xi^{11} + \dots$$

$$I_1(\xi) = 0.64790 - 0.761796\xi^2 + 1.205206\xi^4 - 2.001326\xi^6 + 3.398026\xi^8 - 5.887926\xi^{10} + \dots$$

$$I_2(\xi) = 0.292344 - 0.317691\xi^2 + 0.445267\xi^4 - 0.631968\xi^6 + 0.887044\xi^8 - 1.232238\xi^{10} + \dots$$

Now the parameters required are given in terms of these quantities as follows.

$$\lambda = \frac{\delta_2^2}{\nu}\frac{du_1}{dx} = (1+3\xi^2)\,I_2^2$$

$$l = \frac{\delta_2}{u_1}\left(\frac{\partial u}{\partial y}\right)_w = \frac{TI_2}{(\xi+\xi^3)}$$

$$H = \frac{\delta_1}{\delta_2} = \frac{I_1}{I_2}$$

$$L = 2\left\{1-\lambda(H+2)\right\}$$

The series expansions are

$$\lambda = 0.085465 + 0.070645\xi^2 - 0.195980\xi^4 + 0.431393\xi^6 - 0.838807\xi^8 + 1.508467\xi^{10} + \ldots$$

$$1 = 0.360340 + 0.095229\xi^2 - 0.257948\xi^4 + 0.559969\xi^6 - 1.088940\xi^8 + 1.994552\xi^{10} + \ldots$$

$$H = 2.216225\left\{1 - 0.089089\xi^2 + 0.240266\xi^4 - 0.530428\xi^6 + 1.052361\xi^8 - 2.106355\xi^{10} \ldots\right.$$

and recasting to give $\xi^2$ in terms of $\lambda$ we have

$$\xi^2 = Z + 2.774152Z^2 + 9.285363Z^3 + 33.920339Z^4 + 130.435153Z^5 + \ldots$$

$$\text{where } Z = \frac{\lambda - 0.085465}{0.070645}$$

Elimination of $\xi^2$ from 1 gives

$$1(\lambda) = 0.360340 + 0.095229Z + 0.006232Z^2 + 0.013032Z^3 + 0.0261562Z^4 + 0.070900Z^5 + \ldots$$

Elimination of $\xi^2$ from H gives

$$H(\lambda) = 2.216225 - 0.19744Z - 0.015248Z^2 - 0.054481Z^3 - 0.161893Z^4 - 0.871555Z^5 + \ldots$$

## IV.1 The Analysis for $H(\lambda)$

If we denote $H_4(\lambda) = 2.216225 - 0.19744Z - 0.015248Z^2 - 0.054481Z^3$

$$H_5(\lambda) = H_4(\lambda) - 0.161893Z^4$$

$$H_6(\lambda) = H_5(\lambda) - 0.871555Z^5$$

where $Z = (\lambda - 0.085465)/0.070645$ then

| $\lambda$ | $H_4(\lambda)$ | $H_5(\lambda)$ | $H_6(\lambda)$ |
|---|---|---|---|
| 0.0855 | 2.21613 | 2.21613 | 2.21613 |
| 0.0860 | 2.21473 | 2.21473 | 2.21473 |
| 0.0870 | 2.21193 | 2.21193 | 2.21193 |
| 0.0880 | 2.20912 | 2.20912 | 2.20912 |
| 0.0890 | 2.20630 | 2.20630 | 2.20630 |
| 0.0900 | 2.20347 | 2.20347 | 2.20347 |
| 0.0910 | 2.20063 | 2.20063 | 2.20063 |
| 0.0920 | 2.19779 | 2.19778 | 2.19777 |
| 0.0930 | 2.19493 | 2.19491 | 2.19489 |
| 0.0940 | 2.19205 | 2.19202 | 2.19200 |
| 0.0950 | 2.18916 | 2.18911 | 2.18907 |
| 0.0960 | 2.18626 | 2.18618 | 2.18612 |
| 0.0970 | 2.18334 | 2.18323 | 2.18313 |
| 0.0980 | 2.18041 | 2.18025 | 2.18009 |

Check.

When $\lambda = 0.0960$, $Z = 0.149126$

$H = 2.216225 - 0.029443 - 0.000339 - 0.000181 - 0.000080 - 0.000064 = 2.2186118.$

These series give the following limiting values for H, corresponding to $\lambda = 0.098427$.

$H_4\ (\lambda = 0.098427) = 2.179150$; $H_5(\lambda = 0.098427) = 2.1789(6)$

$H_6\ (\lambda = 0.098427) = 2.1787(8)$

It is known that as $\xi \to \infty$ the solution tends to the Falkner-Skan solution for $u_1 = u_0\xi^3$. Thus the correct limiting value for H is $H = 2.177858$. We may therefore confidently assume that the above procedure yields H over the whole significant range of $\lambda$ to four figures. To see if we can increase this accuracy we make use of the following idea. We assume in turn that the main contributions to H come from the first four, first five and first six terms and that the remaining terms can be absorbed into one term which respresents a small correction we can obtain from $H_4(\lambda)$, $H_5(\lambda)$ and $H_6(\lambda)$ new series expansions.

$$H_5^{(c)} = H_4 + \alpha_1 Z^4; \; H_6^{(c)} = H_5 + \beta_1 Z^5; \; H_7^{(c)} = H_6 + \gamma_1 Z^6$$

where $\alpha_1$, $\beta_1$ and $\gamma_1$ are coefficients chosen to give the correct limiting value of H corresponding to limiting value of $\lambda = 0.098427$. The required values of $\alpha_1$, $\beta_1$ and $\gamma_1$ are

$$\alpha_1 = -1.140033; \; \beta_1 = -5.332032; \; \gamma_1 = -24.305722$$

The results using these series are

| $\lambda$ | $H_5^{(c)}$ | $H_6^{(c)}$ | $H_7^{(c)}$ |
|---|---|---|---|
| 0.086 | 2.214728 | 2.214728 | 2.214728 |
| 0.088 | 2.209115 | 2.209115 | 2.209116 |
| 0.090 | 2.203452 | 2.203463 | 2.203466 |
| 0.092 | 2.197701 | 2.197737 | 2.197751 |
| 0.094 | 2.191808 | 2.191878 | 2.191916 |
| 0.096 | 2.185696 | 2.185786 | 2.185847 |
| 0.098 | 2.179275 | 2.179306 | 2.179332 |

It is seen that the original series expansions for H, $(H_4(\lambda), H_5(\lambda), H_6(\lambda))$ converge down to a limit, whereas the new 'corrected' series expansions $(H_5^{(c)}(\lambda), H_6^{(c)}(\lambda), H_7^{(c)}(\lambda))$ appear to converge up to a limit. A new series in each case was constructed between the four term and 'corrected' five term, five term and corrected six term, and six term and corrected seven term. The difference between old and corrected series was divided according to the ratio

$$\frac{H_n - H_{n+1}}{-H_{n+1}^{(c)} + H_{n+2}^{(c)}} : \frac{H_{n+1} - H_{n+2}}{-H_{n+2}^{(c)} + H_{n+3}^{(c)}}$$

which was found to be approximately $\frac{-1}{3}$. The new series expansions were taken to be

$$H_1^{(c)} = H_5^{(c)} + \alpha(H_5^{(c)} - H_4)$$
$$H_2^{(c)} = H_6^{(c)} + \alpha(H_6^{(c)} - H_5)$$
$$H_3^{(c)} = H_7^{(c)} + \alpha(H_7^{(c)} - H_6)$$

With the value of $\alpha$ taken to be $\frac{-1}{3}$, the results for $H_1^{(c)}$, $H_2^{(c)}$ and $H_3^{(c)}$ are

| $\lambda$ | $H_1^{(c)}$ | $H_2^{(c)}$ | $H_3^{(c)}$ |
|---|---|---|---|
| 0.086 | 2.2147284 | 2.2147284 | 2.2147274 |
| 0.087 | 2.2119246 | 2.2119255 | 2.2119246 |
| 0.088 | 2.2091150 | 2.2091150 | 2.2091150 |
| 0.089 | 2.2062931 | 2.2062950 | 2.2062950 |
| 0.090 | 2.2034578 | 2.2034645 | 2.2034664 |
| 0.091 | 2.2006044 | 2.2006159 | 2.2006197 |
| 0.092 | 2.1977291 | 2.1977482 | 2.1977558 |
| 0.093 | 2.1948261 | 2.1948538 | 2.1948671 |
| 0.094 | 2.1918879 | 2.1919231 | 2.1919413 |
| 0.095 | 2.1889105 | 2.1889486 | 2.1889706 |
| 0.096 | 2.1858835 | 2.1859169 | 2.1859360 |
| 0.097 | 2.1828012 | 2.1828136 | 2.1828175 |
| 0.098 | 2.1796513 | 2.1796179 | 2.1795845 |

$H_1^{(c)}$, $H_2^{(c)}$, $H_3^{(c)}$ are now regarded as three successive estimates for H. Under this assumption the final table for $H(\lambda)$ was calculated using the formula

$$H(\lambda) = H_3^{(c)} + \frac{(H_3^{(c)} - H_2^{(c)})^2}{(2H_2^{(c)} - H_1^{(c)} - H_3^{(c)})}$$

| $\lambda$ | $H(\lambda)$ |
|---|---|
| 0.086 | 2.2147284 |
| 0.087 | 2.2119246 |
| 0.088 | 2.2091150 |
| 0.089 | 2.2062950 |
| 0.090 | 2.2034664 |
| 0.091 | 2.2006216 |
| 0.092 | 2.1977606 |
| 0.093 | 2.1948795 |
| 0.094 | 2.1919603 |
| 0.095 | 2.1890001 |
| 0.096 | 2.1859608 |
| 0.097 | 2.1828184 |
| 0.098 | 2.1795845 |

IV.2 The Analysis for $l(\lambda)$

The calculated series for $l(\lambda)$ is

$$l(\lambda) = 0.360340 + 0.095229Z + 0.006232Z^2 + 0.013032Z^3 + 0.0261562Z^4 + + 0.070900Z^5 + \ldots$$

If we denote by $l_4(\lambda) = 0.360340 + 0.095229Z + 0.006232Z^2 + + 0.013032Z^3$

$$l_5(\lambda) = l_4(\lambda) + 0.0261562Z^4$$

$$l_6(\lambda) = l_5(\lambda) + 0.070900Z^5$$

then we have

| $\lambda$ | $l_4(\lambda)$ | $l_5(\lambda)$ | $l_6(\lambda)$ |
|---|---|---|---|
| 0.086 | 0.361061 | 0.361061 | 0.361061 |
| 0.088 | 0.363766 | 0.363766 | 0.363766 |
| 0.090 | 0.366482 | 0.366482 | 0.366483 |
| 0.092 | 0.369213 | 0.369214 | 0.369215 |
| 0.094 | 0.371959 | 0.371964 | 0.371966 |
| 0.096 | 0.374723 | 0.374736 | 0.374741 |
| 0.098 | 0.377506 | 0.377532 | 0.377544 |

These series expansions give the following limiting values for l, corresponding to $\lambda = 0.098427$.

$l_4(0.098427) = 0.37810(3)$; $l_5(0.098427) = 0.37813(3)$;

$l_6(0.098427) = 0.37814(7)$

The correct limiting value of l is 0.378403 when $\lambda = 0.098427$ or $Z = 0.1834807$. If we therefore assume in turn that the first four, first five and first six terms give the main contribution to l and that the remaining terms can be absorbed into one term which represents a small correction, we can obtain from $l_4(\lambda)$, $l_5(\lambda)$ and $l_6(\lambda)$ new series expansions.

$$l_5^{(c)}(\lambda) = l_4(\lambda) + \alpha_2 Z^4;\ l_6^{(c)}(\lambda) = l_5(\lambda) + \beta_2 Z^5;\ l_7^{(c)}(\lambda) = l_6(\lambda) + \gamma_2 Z^6$$

where $\alpha_2$, $\beta_2$ and $\gamma_2$ are coefficients chosen to give the correct limiting value of l corresponding to the limiting value of $\lambda$ which is $\lambda = 0.098427$.

The required values of $\alpha_2$, $\beta_2$ and $\gamma_2$ are

$$\alpha_2 = 0.264714;\ \beta_2 = 1.300425;\ \gamma_2 = 6.699767$$

The results given by these series are

| $\lambda$ | $l_5^{(c)}(\lambda)$ | $l_6^{(c)}(\lambda)$ | $l_7^{(c)}(\lambda)$ |
|---|---|---|---|
| 0.0860 | 0.361061 | 0.361061 | 0.361061 |
| 0.0880 | 0.363766 | 0.363766 | 0.363766 |
| 0.0900 | 0.366486 | 0.366484 | 0.366483 |
| 0.0920 | 0.369232 | 0.369223 | 0.369219 |
| 0.0940 | 0.372015 | 0.371998 | 0.371987 |
| 0.0960 | 0.374854 | 0.374831 | 0.374814 |
| 0.0980 | 0.377768 | 0.377760 | 0.377753 |

An examination of the tables of $l_4$, $l_5$, $l_6$ and of $l_5^{(c)}$, $l_6^{(c)}$ and $l_7^{(c)}$ indicates that the former estimates seem to be converging up to a limit while the latter are converging down to a limit. A new series in each case was constructed between the four term and 'corrected' five term, five term and 'corrected' six term, and six term and 'corrected' seven term by the formulae

$$l_1^{(c)}(\lambda) = \beta l_4(\lambda) + (1 - \beta) l_5^{(c)}(\lambda)$$

$$l_2^{(c)}(\lambda) = \beta l_5(\lambda) + (1 - \beta) l_6^{(c)}(\lambda)$$

$$l_3^{(c)}(\lambda) = \beta l_6(\lambda) + (1 - \beta) l_7^{(c)}(\lambda)$$

where by reasoning, similar to that for $H(\lambda)$, $\beta$ was chosen equal to 0.4. This yielded the table.

| $\lambda$ | $l_1^{(c)}(\lambda)$ | $l_2^{(c)}(\lambda)$ | $l_3^{(c)}(\lambda)$ |
|---|---|---|---|
| 0.086 | 0.361061 | 0.361061 | 0.361061 |
| 0.087 | 0.362412 | 0.362412 | 0.362412 |
| 0.088 | 0.363766 | 0.363766 | 0.363766 |
| 0.089 | 0.365123 | 0.365122 | 0.365122 |
| 0.090 | 0.366485 | 0.366483 | 0.366483 |
| 0.091 | 0.367851 | 0.367849 | 0.367848 |
| 0.092 | 0.369224 | 0.369220 | 0.369217 |
| 0.093 | 0.370604 | 0.370598 | 0.370594 |
| 0.094 | 0.371993 | 0.371984 | 0.371979 |
| 0.095 | 0.373391 | 0.373382 | 0.373375 |
| 0.096 | 0.374801 | 0.374793 | 0.374785 |
| 0.097 | 0.376224 | 0.376221 | 0.376215 |
| 0.098 | 0.377663 | 0.377669 | 0.377670 |

These l values are now regarded as three successive estimates of the value of $l(\lambda)$ at each stage. Under the assumption that the behaviour is geometric the final table for $l(\lambda)$ may be calculated from

$$l(\lambda) = l_3^{(c)}(\lambda) + \frac{(l_3^{(c)}(\lambda) - l_2^{(c)}(\lambda))^2}{(2l_2^{(c)}(\lambda) - l_3^{(c)}(\lambda) - l_1^{(c)}(\lambda))}$$

and is

| $\lambda$ | $l(\lambda)$ |
|---|---|
| 0.0860 | 0.361061 |
| 0.0870 | 0.362412 |
| 0.0880 | 0.363766 |
| 0.0890 | 0.365122 |
| 0.0900 | 0.366483 |
| 0.0910 | 0.367847 |
| 0.0920 | 0.369215 |
| 0.0930 | 0.370588 |
| 0.0940 | 0.371965 |
| 0.0950 | 0.373338 |
| 0.0960 | 0.374785 |
| 0.0970 | 0.376231 |
| 0.0980 | 0.377670 |

Using these tables of $H(\lambda)$ and $l(\lambda)$ and making use of the relationship $L(\lambda) = 2\left\{l - \lambda(H + 2)\right\}$ the following table may be constructed.

| $\lambda$ | $H(\lambda)$ | $l(\lambda)$ | $L(\lambda)$ |
|---|---|---|---|
| 0.086 | 2.21473 | 0.361061 | -0.002811 |
| 0.087 | 2.21192 | 0.362412 | -0.008050 |
| 0.088 | 2.20911 | 0.363766 | -0.013271 |
| 0.089 | 2.20629 | 0.365122 | -0.018475 |
| 0.090 | 2.20347 | 0.366483 | -0.023658 |
| 0.091 | 2.20062 | 0.367847 | -0.028818 |
| 0.092 | 2.19776 | 0.369215 | -0.033957 |
| 0.093 | 2.19488 | 0.370588 | -0.039071 |
| 0.094 | 2.19196 | 0.371965 | -0.044157 |
| 0.095 | 2.18900 | 0.373338 | -0.049233 |
| 0.096 | 2.18596 | 0.374785 | -0.054133 |
| 0.097 | 2.18282 | 0.376231 | -0.059003 |
| 0.098 | 2.17958 | 0.377670 | -0.063796 |

To determine to how many figures we believe our results, we may consider the values of the sets $H_4$, $H_5$, $H_6$; $H_5^{(c)}$, $H_6^{(c)}$, $H_7^{(c)}$; $H_1^{(c)}$, $H_2^{(c)}$, $H_3^{(c)}$; and H at various stations in our tables and similarly for $l_4$, $l_5$, $l_6$; $l_5^{(c)}$, $l_6^{(c)}$, $l_7^{(c)}$; $l_1^{(c)}$, $l_2^{(c)}$, $l_3^{(c)}$; and l.

<u>For H</u> At <u>$\lambda = 0.086$</u> we have

$H_4 = 2.2147284$ $H_5^{(c)} = 2.2147284$ $H_1^{(c)} = 2.2147284$
$H_5 = 2.2147284$ $H_6^{(c)} = 2.2147284$ $H_2^{(c)} = 2.2147284$
$H_6 = 2.2147284$ $H_7^{(c)} = 2.2147274$ $H_3^{(c)} = 2.2147274$

$H = 2.2147284$

At <u>$\lambda = 0.090$</u>

$H_4 = 2.2034721$ $H_5^{(c)} = 2.2034521$ $H_1^{(c)} = 2.2034578$
$H_5 = 2.2034693$ $H_6^{(c)} = 2.2034626$ $H_2^{(c)} = 2.2034645$
$H_6 = 2.2034683$ $H_7^{(c)} = 2.2034664$ $H_3^{(c)} = 2.2034664$

$H = 2.2034664$

At <u>$\lambda = 0.094$</u>

$H_4 = 2.1920509$ $H_5^{(c)} = 2.1918077$ $H_1^{(c)} = 2.1918879$
$H_5 = 2.1920156$ $H_6^{(c)} = 2.1918783$ $H_2^{(c)} = 2.1919231$
$H_6 = 2.1919928$ $H_7^{(c)} = 2.1919165$ $H_3^{(c)} = 2.1919413$

$H = 2.1919603$

At <u>$\lambda = 0.098$</u>

$H_4 = 2.1804056$ $H_5^{(c)} = 2.1792755$ $H_1^{(c)} = 2.1796513$
$H_5 = 2.1802444$ $H_6^{(c)} = 2.1793060$ $H_2^{(c)} = 2.1796179$
$H_6 = 2.1800909$ $H_7^{(c)} = 2.1793318$ $H_3^{(c)} = 2.1795845$

$H = 2.1795845$

At <u>$\lambda = 0.098427$</u>

The value of H to be used is $H = 2.177858$.

For l    At $\lambda = 0.086$ we have

| | | |
|---|---|---|
| $l_4 = 0.36106145$ | $l_5^{(c)} = 0.36106145$ | $l_1^{(c)} = 0.36106145$ |
| $l_5 = 0.36106145$ | $l_6^{(c)} = 0.36106145$ | $l_2^{(c)} = 0.36106145$ |
| $l_6 = 0.36106145$ | $l_7^{(c)} = 0.36106145$ | $l_3^{(c)} = 0.36106145$ |

$l = 0.36106145$

At $\lambda = 0.090$

| | | |
|---|---|---|
| $l_4 = 0.36648208$ | $l_5^{(c)} = 0.36648655$ | $l_1^{(c)} = 0.36648476$ |
| $l_5 = 0.36648250$ | $l_6^{(c)} = 0.36648387$ | $l_2^{(c)} = 0.36648327$ |
| $l_6 = 0.36648256$ | $l_7^{(c)} = 0.36648297$ | $l_3^{(c)} = 0.36648279$ |

$l = 0.36648256$

At $\lambda = 0.094$

| | | |
|---|---|---|
| $l_4 = 0.37195885$ | $l_5^{(c)} = 0.37201524$ | $l_1^{(c)} = 0.37199265$ |
| $l_5 = 0.37196440$ | $l_6^{(c)} = 0.37199783$ | $l_2^{(c)} = 0.37198442$ |
| $l_6 = 0.37196618$ | $l_7^{(c)} = 0.37198699$ | $l_3^{(c)} = 0.37197864$ |

$l = 0.37196493$

At $\lambda = 0.098$

| | | |
|---|---|---|
| $l_4 = 0.37750584$ | $l_5^{(c)} = 0.37776816$ | $l_1^{(c)} = 0.37766320$ |
| $l_5 = 0.37753171$ | $l_6^{(c)} = 0.37776041$ | $l_2^{(c)} = 0.37766892$ |
| $l_6 = 0.37754416$ | $l_7^{(c)} = 0.37775320$ | $l_3^{(c)} = 0.37766957$ |

$l = 0.37766963$

At $\lambda = 0.098427$

The value of l to be used is $l = 0.378403$.

From these sample calculations, in unrounded form we see that we may expect our values of H to be correct to at least 4 decimal places over the whole range and our values of l to at least 5 decimal places over the whole range.

Section V: Calculation of parameters H, l, L, $\lambda$ and $\mu$ for flow $u_1 = u_o(1 + \xi)$

From experience gained in calculating the same parameters for the flow $u_1 = u_o(\xi + \xi^3)$ it was decided to calculate H and l as functions of $\lambda$, and not to try to calculate for the whole range of $\xi$ the parameters H, l, L and $\mu$, as functions of $\xi$. For small values of $\xi$, H, l and $\lambda$ were calculated using Howarth-Blasius series techniques. Then $\xi$ was eliminated between $\lambda$ and H, and between $\lambda$ and l to give $H(\lambda)$ and $l(\lambda)$ respectively. The function L was calculated from $L = 2\left\{1 - \lambda(H + 2)\right\}$. Once a tabulation of L v $\lambda$ was obtained an attempt to tabulate $\xi$ against $\lambda$ was made by solving the momentum integral equation by Runge Kutta techniques, which shall be described in Section VI.

Since H, l, $\lambda$, L and $\mu$ involve skin friction, displacement thickness and momentum thickness, these three quantities were calculated first of all.

Skin friction

If we follow Howarth's notation (10), let

$$U_1 = b_o + b_1 x$$

where $b_o$ and $b_1$ are positive constants, and assume an expansion of the form

$$\psi = b_o^{\frac{1}{2}} x^{\frac{1}{2}} \nu^{\frac{1}{2}} \Big\{ f_o(\eta) + (8x^*) f_1(\eta) + (8x^*)^2 f_2(\eta) + (8x^*)^3 f_3(\eta) + (8x^*)^4 f_4(\eta) + (8x^*)^5 f_5(\eta) + (8x^*)^6 f_6(\eta) + \dots$$

where $$\eta = \tfrac{1}{2} y x^{-\frac{1}{2}} \nu^{-\frac{1}{2}} b_o^{\frac{1}{2}} \text{ and } x^{*} = \frac{b_1 x}{b_o}$$

Now on defining $b_o = u_o$, $b_1 = \frac{u_o}{c}$ and $\xi = x^{*} = \frac{x}{c}$ where c is some length characteristic of distance tangential to the surface we find

$$\left(\frac{\partial u}{\partial y}\right)_w = \frac{b_o}{2}\left\{ f_o''(0) + (8\xi) f_1''(0) + \ldots\ldots \right\} \frac{\partial \eta}{\partial y}$$

$$= \frac{b_o}{2} \cdot \tfrac{1}{2}\, x^{-\frac{1}{2}} \nu^{-\frac{1}{2}} b_o^{\frac{1}{2}} \left\{ f_o''(0) + (8\xi) f_1''(0) + \ldots \right.$$

$$= \frac{1}{4} \left(\frac{u_o^3}{\nu c}\right)^{\frac{1}{2}} \left(\frac{x}{c}\right)^{-\frac{1}{2}} \left\{ f_o''(0) + (8\xi) f_1''(0) + \ldots \right.$$

$$\left(\frac{\nu c}{u_o^3}\right)^{\frac{1}{2}} \left(\frac{\partial u}{\partial y}\right)_w = \frac{1}{4} (\xi)^{-\frac{1}{2}} \left\{ f_o''(0) + (8\xi) f_1''(0) + (8\xi)^2 f_2''(0) + \ldots \right.$$

From Howarth's paper we have that

$$f_o''(0) = 1.328242;\ f_1''(0) = 1.02054;\ f_2''(0) = -0.06926;\ f_3''(0) = 0.0560$$

$$f_4''(0) = -0.0372;\ f_5''(0) = 0.0272;\ f_6''(0) = -0.0212;\ f_7''(0) = 0.0174$$

$$f_8''(0) = -0.0147$$

$$\therefore \left(\frac{\nu c}{u_o^3}\right)^{\frac{1}{2}} \left(\frac{\partial u}{\partial y}\right)_w$$

$$= 0.33206\xi^{-\frac{1}{2}} + 2.04108\xi^{\frac{1}{2}} - 1.10816\xi^{3/2} + 7.168\xi^{5/2} - 38.0928\xi^{7/2} + 222.8224\xi^{9/2} - 1389.3632\xi^{11/2} + 9{,}122.6112\xi^{13/2} + \ldots\ldots$$

<u>Displacement Thickness</u>

$$\delta_1 = \int_0^\infty \left(1 - \frac{u}{u_1}\right) dy.\quad dy = 2\nu^{\frac{1}{2}} u_o^{-\frac{1}{2}} x^{\frac{1}{2}}\, d\eta$$

$$u = \frac{u_o}{2}\left\{ f_o'(\eta) + (8\xi) f_1'(\eta) + (8\xi)^2 f_2'(\eta) + (8\xi)^3 f_3'(\eta) + \ldots \right.$$

$$u_1 = u_o(1 + \xi)$$

On performing the algebra we have that

$$\left(\frac{u_o}{\nu c}\right)^{\frac{1}{2}} \delta_1 = \frac{a_o\xi^{\frac{1}{2}} + a_1\xi^{3/2} + a_2\xi^{5/2} + a_3\xi^{7/2} + a_4\xi^{9/2} + a_5\xi^{11/2} + \ldots}{1 + \xi}$$

where
$$a_\circ = \lim_{\eta\to\infty} (2\eta - f_\circ) = 1.72077$$
$$a_1 = \lim_{\eta\to\infty} (2\eta - 8f_1) = 2.75560$$
$$a_2 = -\lim_{\eta\to\infty} 8^2 f_2 = 12.1088$$
$$a_3 = -\lim_{\eta\to\infty} 8^3 f_3 = -58.0096$$
$$a_4 = -\lim_{\eta\to\infty} 8^4 f_4 = 315.392$$
$$a_5 = -\lim_{\eta\to\infty} 8^5 f_5 = -1877.6064$$

$$\therefore \left(\frac{u_\circ}{\nu c}\right)^{\frac{1}{2}} \delta_1 = \frac{1.72077\xi^{\frac{1}{2}} - 2.75560\xi^{3/2} + 12.1088\xi^{5/2} - 58.0096\xi^{7/2} + 315.392\xi^{9/2} - 1877.6064\xi^{11/2}}{(1+\xi)}$$

$$= 1.72077\xi^{\frac{1}{2}} - 4.47637\xi^{3/2} + 16.58517\xi^{5/2} - 74.59477\xi^{7/2} + 389.98677\xi^{9/2} - 2267.59317\xi^{11/2} + \ldots\ldots$$

Momentum Thickness

If we denote

$$I_1 = \frac{1.72077\xi^{\frac{1}{2}} - 2.755600\xi^{3/2} + 12.108800\xi^{5/2} - 58.009600\xi^{7/2} + 315392\xi^{9/2} - 1877.6064\xi^{11/2}}{(1+\xi)}$$

$$T = 0.33206\xi^{-\frac{1}{2}} + 2.041080\xi^{\frac{1}{2}} - 1.108160\xi^{3/2} + 7.168\xi^{5/2} - 38.0928\xi^{7/2} + 222.8224\xi^{9/2} +$$

the momentum integral equation is, where we define $I_2 = \left(\frac{u_\circ}{\nu c}\right)^{\frac{1}{2}} \delta_2$,

$$\frac{d}{d\xi}\left[(1+\xi)^2 I_2\right] = T - I_1(1+\xi).$$

This yields

$$I_2(\xi) = \frac{0.664120\xi^{\frac{1}{2}} + 0.213540\xi^{3/2} + 0.658976\xi^{5/2} - 1.411657\xi^{7/2} + 4.425956\xi^{9/2} - 16.830836\xi^{11/2} + \ldots}{(1+\xi)^2}$$

$$= 0.664120\xi^{\frac{1}{2}}-1.114700\xi^{3/2}+2.224256\xi^{5/2}-4.745469\xi^{7/2}+11.692638\xi^{9/2}$$
$$-35.470643\xi^{11/2}+ \ldots\ldots$$

The Parameters $\lambda$, H and l.

Calculation of $\lambda$

$$\lambda = \frac{\delta_2^2}{\nu}\frac{du_1}{dx} = I_2^2 \text{ (in above notation)}$$
$$= (0.664120\xi^{\frac{1}{2}}-1.1147\xi^{3/2}+2.224256\xi^{5/2}-4.745469\xi^{7/2}+11.692638\xi^{9/2}$$
$$-35.470643\xi^{11/2})^2$$

$$\therefore \quad = 0.441055\xi-1.480589\xi^2+4.196902\xi^3-11.261878\xi^4+31.057493\xi^5$$
$$-94.291370\xi^6+ \ldots$$

and expressing $\xi$ as a function of $\lambda$ we have

$$\xi = Z+3.356926Z^2+13.022306Z^3+54.963351Z^4+241.519743Z^5+\ldots$$

where $Z = 2.267291\lambda$

Calculation of l

$$l = \frac{\delta_2}{u_1}\left(\frac{\partial u}{\partial y}\right)_w = \frac{TI_2}{1+\xi}$$

$$= 0.220528+0.764847\xi-3.037404\xi^2+11.997186\xi^3-53.553602\xi^4+277.285892\xi^5+$$

$\therefore$ in terms of $\lambda$ (using $Z = 2.267291\lambda$) we have

$$l(\lambda) = 0.220528+0.764847Z-0.469869Z^2+1.564577Z^3-4.030419Z^4+17.738310Z^5+\ldots$$

Calculation of H

$$H = \frac{\delta_1}{\delta_2} = \frac{I_1}{I_2}$$

$$= \frac{1.72077\xi^{\frac{1}{2}}-4.47637\xi^{3/2}+16.58517\xi^{5/2}-74.59477\xi^{7/2}+389.98677\xi^{9/2}-2267.59317\xi^{11/2}+\ldots}{0.664120\xi^{\frac{1}{2}}-1.114700\xi^{3/2}+2.224256\xi^{5/2}-4.745469\xi^{7/2}+11.692638\xi^{9/2}-35.470643\xi^{11/2}+\ldots}$$

$$= 2.591053\left\{\frac{1.-2.601376\xi+9.638226\xi^2-43.349646\xi^3+226.635035\xi^4-1317.77818\xi^5+}{1-1.678462\xi+3.349178\xi^2-7.14549\xi^3+17.606212\xi^4-53.40999\xi^5+\ \ldots.}\right.$$

$$= 2.591053\left\{1-0.922914\xi+4.739972\xi^2-25.157281\xi^3+144.333593\xi^4-887.735049\xi^5+.\right.$$

$$= 2.591053-2.391319\xi+12.281519\xi^2-65.183848\xi^3+373.975989\xi^4-2300.168562\xi^5+..$$

In terms of $\lambda$, this becomes

$$H(\lambda) = 2.591053-2.391319Z+4.254037Z^2-13.868038Z^3+44.356244Z^4-182.435479Z^5+.$$

To obtain the final forms for $H(\lambda)$ and $l(\lambda)$ and hence also of $L(\lambda)$ the following procedure was adopted.

For $H(\lambda)$ the four, five and six term estimates were defined by

$$H_4(\lambda) = 2.591053-2.391319Z+4.254037Z^2-13.868038Z^3$$

$$H_5(\lambda) = H_4(\lambda) + 44.356244Z^4$$

$$H_6(\lambda) = H_5(\lambda) - 182.435479Z^5$$

For $l(\lambda)$, three similar functions were defined by

$$l_4(\lambda) = 0.220528 + 0.764847Z - 0.469869Z^2 + 1.564577Z^3$$

$$l_5(\lambda) = l_4(\lambda) - 4.030419Z^4$$

$$l_6(\lambda) = l_5(\lambda) + 17.738310Z^5$$

It was found that the estimates for $H(\lambda)$ yielded values which oscillated with decreasing oscillation as the number of terms was increased. (A similar result was obtained for $l(\lambda)$.) It was therefore thought that in each case the effect of the remaining terms in the series could be assessed by assuming that these remaining terms obeyed a geometric progression in which the common ratio was indicated by considering the final terms in the four, five and six term expansions in each case. The correction due to the remaining terms could then be expressed as the sum of a geometric progression so that new estimates for $H(\lambda)$ and $l(\lambda)$ could be defined by

$$H^*(\lambda) = H_6(\lambda) + \Delta H \text{ where } \Delta H = \frac{(H_6(\lambda) - H_5(\lambda))^2}{(2H_5(\lambda) - H_4(\lambda) - H_6(\lambda))}$$

$$l^*(\lambda) = l_6(\lambda) + \Delta l \text{ where } \Delta l = \frac{(l_6(\lambda) - l_5(\lambda))^2}{(2l_5(\lambda) - l_4(\lambda) - l_6(\lambda))}$$

The results for these estimates for $H(\lambda)$ and $l(\lambda)$ are given in tables 2.V.1 and 2.V.3 respectively. From table 2.V.1 it is seen that for $0 \leqslant \lambda \leqslant 0.060$, $\Delta H \leqslant 0.003$, a small correction, but that as $\lambda$ increases beyond this range, the values for $\Delta H$ increase rapidly. Corresponding to the limiting value of $\lambda = 0.085465$ the value of $\Delta H$ is 0.0221 which gives a value for $H^*(\lambda)$ of 2.22131. To make our estimate for H, $H^*$, agree with the correct limiting value of H at this point, which is 2.21622(5) the correction should only have been 0.01702. That our correction term $\Delta H$ does slightly overestimate this value at the limiting case is not too surprising since the assumption that the terms are geometric is not strictly true, especially in the range of $\lambda$ near the limiting value. A minor adjustment to these correction terms to give the correct limiting value is made by defining $\Delta H' = \beta \Delta H = \frac{0.01702}{0.02201} \times \Delta H = 0.7701 \Delta H$ and taking our final estimate for $H(\lambda)$ to be

$$H(\lambda) = H_6(\lambda) + 0.7701 \Delta H.$$

This changes the earlier values in the table by a very small amount but should give more reliable results near the limiting case. These results are given in table 2.V.2. A similar analysis may be carried out for the estimates for $l(\lambda)$. In this case, by reasoning analogous to that above we define $\Delta l' = \frac{-0.00130}{-0.00223}\Delta l = 0.5837\Delta l$ and define our final estimate for $l(\lambda)$ to be

$$l(\lambda) = l_6(\lambda) + 0.5837\Delta l$$

The results given by this process are given in Table 2.V.4.

We can easily see that by altering these values of $\Delta H$ and $\Delta l$ (which at worst represent changes in H and l of order 1% and 0.6% respectively) there is very little change in H and l apart from the few values near the limiting case.

For H we see that for $\lambda \leq 0.020$, $\Delta H$ is zero to 5 decimal places; $\Delta H'$ is therefore also zero. At a value typical of the middle of the range e.g. $\lambda = 0.045$ we see that $\Delta H = 0.00059$ has been replaced by $\Delta H' = 0.00046$ so that the difference $\Delta H - \Delta H' = 0.00013$. The percentage change in the value of H, (=2.4010) at this point, by altering $\Delta H$ to $\Delta H'$ is therefore of order 0.006. At a value nearer the limiting case e.g. $\lambda = 0.065$, we have that $\Delta H$ and $\Delta H'$ differ by 0.0011 giving the percentage change in H to be 0.05.

A similar situation occurs with the values for l. For $\lambda \leq 0.03$, $\Delta l$, and therefore $\Delta l'$, is zero to five decimal places. For $\lambda = 0.045$, the change, $\Delta l' - \Delta l = 0.0002$, giving a percentage change in l of order 0.06. These changes are very small and show that this last approximation (replacing $\Delta H$ by $\Delta H'$, $\Delta l$ by $\Delta l'$) introduces very small errors.

The final values used for $H(\lambda)$, $l(\lambda)$ and $L(\lambda)$ in further calculations are given in table 2.V.5.

Calculation for H

Table 2.V.1.

| $\lambda$ | $H_4(\lambda)$ | $H_5(\lambda)$ | $H_6(\lambda)$ | $H^*(\lambda)$ | $\Delta H$ |
|---|---|---|---|---|---|
| 0.000 | 2.591053 | 2.591053 | 2.591053 | 2.591053 | 0.00000 |
| 0.005 | 2.56447 | 2.56447 | 2.56447 | 2.56447 | 0.00000 |
| 0.010 | 2.53886 | 2.53887 | 2.53887 | 2.53887 | 0.00000 |
| 0.015 | 2.51410 | 2.51416 | 2.51415 | 2.51415 | 0.00000 |
| 0.020 | 2.49007 | 2.49026 | 2.49022 | 2.49023 | 0.00000 |
| 0.025 | 2.46665 | 2.46711 | 2.46700 | 2.46702 | 0.00002 |
| 0.030 | 2.44371 | 2.44466 | 2.44440 | 2.44445 | 0.00005 |
| 0.035 | 2.42115 | 2.42291 | 2.42233 | 2.42247 | 0.00014 |
| 0.040 | 2.39882 | 2.40182 | 2.40071 | 2.40101 | 0.00030 |
| 0.045 | 2.37662 | 2.38143 | 2.37941 | 2.38001 | 0.00059 |
| 0 050 | 2.35443 | 2.36175 | 2.35834 | 2.35942 | 0.00108 |
| 0.055 | 2.33211 | 2.34284 | 2.33734 | 2.33920 | 0.00186 |
| 0.060 | 2.30956 | 2.32475 | 2.31625 | 2.31930 | 0.00305 |
| 0.065 | 2.28664 | 2.30756 | 2.29488 | 2.29967 | 0.00479 |
| 0.070 | 2.26324 | 2.29138 | 2.27301 | 2.28027 | 0.00726 |
| 0.075 | 2.23924 | 2.27632 | 2.25038 | 2.26106 | 0.0107 |
| 0.080 | 2.21451 | 2.26252 | 2.22670 | 2.24200 | 0.0153 |
| 0.085 | 2.18893 | 2.25012 | 2.20162 | 2.22306 | 0.0214 |
| 0.085465 | 2.18651 | 2.24904 | 2.19920 | 2.22131 | 0.0221 |

Table 2.V.2.

| $\lambda$ | $H(\lambda)$ | $\Delta H'$ | $\lambda$ | $H(\lambda)$ | $\Delta H'$ |
|---|---|---|---|---|---|
| 0.000 | 2.591053 | 0.00000 | 0.060 | 2.31860 | 0.00235 |
| 0.005 | 2.56447 | 0.00000 | 0.065 | 2.29860 | 0.00368 |
| 0.010 | 2.53887 | 0.00000 | 0 070 | 2.27860 | 0.00559 |
| 0.015 | 2.51415 | 0.00000 | 0.075 | 2.25861 | 0.00822 |
| 0.020 | 2.49023 | 0.00000 | 0.080 | 2.23849 | 0.01178 |
| 0.025 | 2.46702 | 0.00001 | 0.085 | 2.21813 | 0.01651 |
| 0.030 | 2.44444 | 0.00004 | 0.085465 | 2.21622(6) | 0.01702 |
| 0.035 | 2.42244 | 0.00011 | | | |
| 0.040 | 2.40094 | 0.00023 | | | |
| 0.045 | 2.37987 | 0.00046 | | | |
| 0.050 | 2.35917 | 0.00084 | | | |
| 0.055 | 2.33877 | 0.00144 | | | |

Calculation for l

Table 2.V.3.

| $\lambda$ | $l_4(\lambda)$ | $l_5(\lambda)$ | $l_6(\lambda)$ | $l^*(\lambda)$ | $\Delta l$ |
|---|---|---|---|---|---|
| 0.000 | 0.220528 | 0.220528 | 0.220528 | 0.220528 | 0.00000 |
| 0.005 | 0.229140 | 0.229140 | 0.229140 | 0.229140 | 0.00000 |
| 0.010 | 0.237646 | 0.237645 | 0.237645 | 0.237645 | 0.00000 |
| 0.015 | 0.246058 | 0.246053 | 0.246053 | 0.246053 | 0.00000 |
| 0.020 | 0.254390 | 0.254373 | 0.254377 | 0.254376 | 0.00000 |
| 0.025 | 0.262656 | 0.262615 | 0.262625 | 0.262623 | 0.00000 |
| 0.030 | 0.270870 | 0.270784 | 0.270810 | 0.270804 | 0.00000 |
| 0.035 | 0.279045 | 0.278886 | 0.278941 | 0.278927 | -0.00001 |
| 0.040 | 0.287196 | 0.286923 | 0.287032 | 0.287001 | -0.00003 |
| 0.045 | 0.295334 | 0.294898 | 0.295094 | 0.295033 | -0.00006 |
| 0.050 | 0.303475 | 0.302810 | 0.303142 | 0.303031 | -0.00011 |
| 0.055 | 0.311632 | 0.310658 | 0.311193 | 0.311003 | -0.00019 |
| 0.060 | 0.319819 | 0.318439 | 0.319265 | 0.318956 | -0.00031 |
| 0.065 | 0.328049 | 0.326148 | 0.327381 | 0.326896 | -0.00048 |
| 0.070 | 0.336336 | 0.333779 | 0.335565 | 0.334831 | -0.00073 |
| 0.075 | 0.344694 | 0.341324 | 0.343846 | 0.342767 | -0.00108 |
| 0.080 | 0.353136 | 0.348774 | 0.352256 | 0.350710 | -0.00154 |
| 0.085 | 0.361677 | 0.356117 | 0.360832 | 0.358668 | -0.00216 |
| 0.085465 | 0.362476 | 0.356794 | 0.361640 | 0.359409 | -0.00223 |

Table 2.V.4.

| $\lambda$ | $l(\lambda)$ | $\Delta l'$ | $\lambda$ | $l(\lambda)$ | $\Delta l'$ |
|---|---|---|---|---|---|
| 0.000 | 0.220528 | 0.00000 | 0.060 | 0.319085 | -0.00018 |
| 0.005 | 0.229140 | 0.00000 | 0.065 | 0.327098 | -0.00028 |
| 0.010 | 0.237645 | 0.00000 | 0.070 | 0.335137 | -0.00043 |
| 0.015 | 0.246053 | 0.00000 | 0.075 | 0.343217 | -0.00063 |
| 0.020 | 0.254376 | 0.00000 | 0.080 | 0.351355 | -0.00090 |
| 0.025 | 0.262624 | 0.00000 | 0.085 | 0.359571 | -0.00126 |
| 0.030 | 0.270806 | 0.00000 | 0.085465 | 0.360340 | -0.00130 |
| 0.035 | 0.278933 | -0.00001 | | | |
| 0.040 | 0.287014 | -0.00002 | | | |
| 0.045 | 0.295058 | -0.00003 | | | |
| 0.050 | 0.303077 | -0.00006 | | | |
| 0.055 | 0.311082 | -0.00011 | | | |

Final Tabulation

Table 2.V.5.

| $\lambda$ | $H(\lambda)$ | $l(\lambda)$ | $L(\lambda)$ |
|---|---|---|---|
| 0.000 | 2.591053 | 0.220528 | 0.441056 |
| 0.005 | 2.56447 | 0.229140 | 0.412636 |
| 0.010 | 2.53887 | 0.237645 | 0.384513 |
| 0.015 | 2.51415 | 0.246053 | 0.356682 |
| 0.020 | 2.49023 | 0.254376 | 0.329144 |
| 0.025 | 2.46702 | 0.262624 | 0.301897 |
| 0.030 | 2.44444 | 0.270806 | 0.274946 |
| 0.035 | 2.42244 | 0.278933 | 0.248295 |
| 0.040 | 2.40094 | 0.287014 | 0.221952 |
| 0.045 | 2.37987 | 0.295058 | 0.195928 |
| 0.050 | 2.35917 | 0.303077 | 0.170237 |
| 0.055 | 2.33877 | 0.311082 | 0.144899 |
| 0.060 | 2.31860 | 0.319085 | 0.119938 |
| 0.065 | 2.29860 | 0.327098 | 0.095383 |
| 0.070 | 2.27860 | 0.335137 | 0.071270 |
| 0.075 | 2.25861 | 0.343217 | 0.047643 |
| 0.080 | 2.23849 | 0.351355 | 0.024552 |
| 0.085 | 2.21813 | 0.359571 | 0.002059 |
| 0.085465 | 2.21622(6) | 0.360340 | 0.000000 |

Section VI: Runge-Kutta techniques applied to the solution of the momentum integral equation for the cases $u_1 = u_o(\xi + \xi^3)$ and $u_1 = u_o(1 + \xi)$

As has been described in Sections III and IV of this chapter, a tabulation of L versus $\lambda$ has been derived for each of the flows $u_1 = u_o(1 + \xi)$ and $u_1 = u_o(\xi + \xi^3)$. Once this has been done the momentum integral equation may be solved to give the value of $\xi$ corresponding to a tabulated value of $\lambda$. This section describes the solution of this equation for both of the cases $u_1 = u_o(1 + \xi)$ and $u_1 = u_o(\xi + \xi^3)$, by a Runge-Kutta type method.

The momentum integral equation is

$\frac{d}{dx}\left(\frac{\lambda}{u_1'}\right) = \frac{L}{u_1}$ where the prime denotes differentation with respect to x.

or $\frac{d}{d\xi}\left(\frac{\lambda}{u_1'}\right) = \frac{L}{u_1}$ where differentiation is now with respect to $\xi = \frac{x}{c}$.

$$\therefore \frac{du_1}{d\xi} \cdot \frac{d\lambda}{d\xi} - \lambda \frac{d^2u_1}{d\xi^2} = \left(\frac{du_1}{d\xi}\right)^2 \frac{L(\lambda)}{u_1(\xi)}$$

$$\therefore \frac{d\lambda}{d\xi} - \lambda \frac{\frac{d^2u_1}{d\xi^2}}{\frac{du_1}{d\xi}} = \left(\frac{du_1}{d\xi}\right) \cdot \frac{L(\lambda)}{u_1(\xi)}$$

$$\therefore \frac{d\lambda}{d\xi} - \lambda g(\xi) = F(\xi)L(\lambda)$$

$$\text{where } g(\xi) = \frac{\frac{d^2u_1}{d\xi^2}}{\frac{du_1}{d\xi}} \text{ and } F(\xi) = \frac{\frac{du_1}{d\xi}}{u_1(\xi)}$$

$$\therefore \frac{d\lambda}{d\xi} = F(\xi)L(\lambda) + \lambda g(\xi)$$

$$\text{or } \frac{d\xi}{d\lambda} = \frac{1}{F(\xi)L(\lambda) + \lambda g(\xi)} = f(\lambda,\xi)$$

*

The Runge-Kutta scheme is then $\xi_{n+1} = \xi_n + hK_4(\xi_n, \lambda_n, h)$ where h is the step-length and $K_4(\xi, \lambda; h) = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$ where

$$k_1 = f(\lambda, \xi);\ k_2 = f(\lambda + \frac{h}{2}, \xi + \frac{h}{2} k_1)$$

$$k_3 = f(\lambda + \frac{h}{2}, \xi + \frac{h}{2} k_2);\ k_4 = f(\lambda + h, \xi + hk_3)$$

The flow $u_1 = u_\circ(1 + \xi)$

For this problem the functions $g(\xi)$ and $F(\xi)$ become

$$g(\xi) \equiv 0;\ F(\xi) = \frac{1}{1+\xi}$$

$\therefore$ (*) reduces to

$$\frac{d\xi}{d\lambda} = \frac{1+\xi}{L(\lambda)}$$

$$\log(1+\xi) = \int_{\lambda_0}^{\lambda} \frac{d\lambda}{L(\lambda)} \text{ where } \lambda = \lambda_0 \text{ when } \xi = 0$$

$$\text{or } \xi = e^{\int_{\lambda_0}^{\lambda} \frac{d\lambda}{L}} - 1$$

This gives the following results

| $\lambda$ | step length 0.005 $\xi$ | step length L = 0.01 $\xi$ |
|---|---|---|
| 0.000 | 0.000000 | 0.000000 |
| 0.005 | 0.011788 | |
| 0.010 | 0.024566 | 0.024574 |
| 0.015 | 0.038491 | |
| 0.020 | 0.053753 | 0.053756 |
| 0.025 | 0.070597 | |
| 0.030 | 0.089335 | 0.089348 |
| 0.035 | 0.110376 | |
| 0.040 | 0.134271 | 0.134287 |
| 0.045 | 0.161783 | |
| 0.050 | 0.194001 | 0.194073 |
| 0.055 | 0.232607 | |
| 0.060 | 0.280217 | 0.280444 |
| 0.065 | 0.341503 | |
| 0.070 | 0.425164 | 0.426496 |
| 0.075 | 0.552528 | |
| 0.080 | 0.793089 | 0.815559 |
| 0.085 | 2.834056 | |

<u>The flow $u_1 = u_0(\xi + \xi^3)$</u>

For this problem the functions $g(\xi)$ and $F(\xi)$ become

$$g(\xi) = \frac{6\xi}{1 + 3\xi^2} \; ; \; F(\xi) = \frac{1 + 3\xi^2}{\xi + \xi^3}$$

For this problem several difficulties were experienced. In the region of $\lambda = 0.0855$, $\frac{d\xi}{d\lambda}$ becomes very large and causes difficulties in the integration procedure. This difficulty could however be removed by using a higher starting value of $\lambda$ away from $\lambda = 0.0855$. The starting value of $\xi$ could be obtained by using the series expansion

$$\xi^2 = Z + 2.774152Z^2 + 9.285363Z^3 + 33.920339Z^4 +$$

$$130.435153Z^5 \text{ where } Z = \left(\frac{\lambda - 0.085465}{0.070645}\right)$$

With a starting value of $\lambda = 0.087$ results were obtained up to $\lambda = 0.095$, using step lengths of 0.001 and 0.002, which were in good agreement with the results which had been obtained previously. In the region $0.095 \leqslant \lambda \leqslant 0.0984$ some more difficulties were encountered. Instead of finding that $\xi$ increased rapidly to infinity as $\lambda$ tended to 0.0984 it was found that $\xi$ remained in the region of 0.6. It was anticipated that at least part of the difficulty lay in the fact that the range $0.6 \leqslant \xi \leqslant \infty$ was represented by the values of $\lambda$ in the range $0.095 \leqslant \lambda \leqslant 0.0984$ and that step lengths of 0.001 and 0.002 were too large in this region. Accordingly the step lengths were reduced to 0.0002 and 0.0001 and the table of values of $L(\lambda)$ to be used, enlarged by means of quadratic interpolation. The reduction in the step length did not, unfortunately, produce a significant improvement in the results. A suggested explanation of the behaviour is as follows. If one considers the parameter $\lambda$ for any Falkner-Skan flow it is seen that this is constant for all $\xi$. The flow $u_1 = u_\circ(\xi + \xi^3)$ may be expected to behave as $u_\circ \xi^3$ for large $\xi$ and so the parameter $\lambda$ may be expected to be roughly constant over a large range. If $\lambda$ increases rapidly from its Hiemenz value to almost the limiting Falkner-Skan value it is to be expected that somewhere in the range the gradient $\frac{d\xi}{d\lambda}$ may be so steep that unless a minute step length were taken the Runge-Kutta method will break down. The results are however presented, though with not a great deal of confidence for $\lambda$ in the region of 0.098. The values of $\mu$, $= \lambda^2 u_1 u_1''/(u_1')^2$, calculated from these results showed an unexpected turning point when plotted against $\xi$. It was later found that this same behaviour was predicted for these parameters using Thwaites' method.

Results

| $\lambda$ | Step length 0.002 $\xi$ | Step length 0.001 $\xi$ | Richardson's extrapolated value |
|---|---|---|---|
| 0.086 | 0.087957 | | |
| 0.087 | 0.152123 | 0.152126 | |
| 0.088 | 0.199874 | 0.199767 | 0.199731 |
| 0.089 | 0.244266 | 0.241610 | 0.240725 |
| 0.090 | 0.280513 | 0.280757 | 0.280838 |
| 0.091 | 0.319418 | 0.318849 | 0.318660 |
| 0.092 | 0.356944 | 0.357023 | 0.357050 |
| 0.093 | 0.396497 | 0.396258 | 0.396179 |
| 0.094 | 0.437643 | 0.437565 | 0.437539 |
| 0.095 | 0.482511 | 0.482291 | 0.482217 |

| $\lambda$ | $\xi$ step length 0.0002 | $\xi$ step length 0.0001 |
|---|---|---|
| 0.0952 | 0.491702 | 0.491702 |
| 0.0954 | 0.501381 | 0.501381 |
| 0.0956 | 0.511272 | 0.511271 |
| 0.0958 | 0.521391 | 0.521390 |
| 0.0960 | 0.531762 | 0.531762 |
| 0.0962 | 0.542418 | 0.542417 |
| 0.0964 | 0.553391 | 0.553391 |
| 0.0966 | 0.564719 | 0.564719 |
| 0.0968 | 0.576438 | 0.576438 |
| 0.0970 | 0.588591 | 0.588591 |
| 0.0972 | 0.601227 | 0.601226 |
| 0.0974 | 0.614400 | 0.614400 |
| 0.0976 | 0.628180 | 0.628179 |
| 0.0978 | 0.642647 | 0.642647 |
| 0.0980 | 0.657902 | 0.657902 |

With these values of $\xi$ and $\lambda$ we obtain the following tabulation for $\mu$

| $\lambda$ | $\mu$ | $\lambda$ | $\mu$ | $\lambda$ | $\mu$ |
|---|---|---|---|---|---|
| 0.086 | 0.000330 | 0.0952 | 0.005484 | 0.0972 | 0.006421 |
| 0.087 | 0.000940 | 0.0954 | 0.005583 | 0.0974 | 0.006509 |
| 0.088 | 0.001537 | 0.0956 | 0.005680 | 0.0976 | 0.006595 |
| 0.089 | 0.002114 | 0.0958 | 0.005776 | 0.0978 | 0.006680 |
| 0.090 | 0.002704 | 0.0960 | 0.005871 | 0.0980 | 0.006764 |
| 0.091 | 0.003265 | 0.0962 | 0.005965 | | |
| 0.092 | 0.003819 | 0.0964 | 0.006058 | | |
| 0.093 | 0.004356 | 0.0966 | 0.006151 | | |
| 0.094 | 0.004879 | 0.0968 | 0.006242 | | |
| 0.095 | 0.005385 | 0.0970 | 0.006332 | | |

The Thwaites' method yields the following results for this problem

$$\delta_2^2 = 0.45\nu u_1^{-6} \int_0^x u_1^5 \, dx \,, \quad \lambda = \frac{u_1^1}{\nu} \delta_2^2$$

$$\lambda = 0.45 \frac{(1+3\xi^2)}{(1+\xi^2)^6}\left[\frac{1}{6} + \frac{5}{8}\xi^2 + \xi^4 + \frac{5}{6}\xi^6 + \frac{5}{14}\xi^8 + \frac{\xi^{10}}{16}\right]$$

$$\mu = \lambda^2 \cdot \frac{6(\xi^2 + \xi^4)}{(1 + 3\xi^2)^2}$$

| $\xi$ | $\lambda$ | $\mu$ |
|---|---|---|
| 0.0 | 0.07500 | 0.00000 |
| 0.1 | 0.07555 | 0.00033 |
| 0.2 | 0.07700 | 0.00118 |
| 0.3 | 0.07894 | 0.00227 |
| 0.4 | 0.08089 | 0.00333 |
| 0.5 | 0.08255 | 0.00417 |
| 0.6 | 0.08380 | 0.00477 |
| 0.7 | 0.08465 | 0.00514 |
| 0.8 | 0.08518 | 0.00536 |
| 0.9 | 0.08548 | 0.00546 |
| 1.0 | 0.08563 | 0.00550 |

| $\xi^{-\frac{1}{2}}$ | $\lambda$ | $\mu$ |
|---|---|---|
| 1.0 | 0.08563 | 0.00550 |
| 0.9 | 0.08566 | 0.00550 |
| 0.8 | 0.08568 | 0.00549 |
| 0.7 | 0.08567 | 0.00547 |
| 0.6 | 0.08562 | 0.00543 |
| 0.5 | 0.08554 | 0.00538 |
| 0.4 | 0.08542 | 0.00530 |
| 0.3 | 0.08525 | 0.00520 |
| 0.2 | 0.08503 | 0.00508 |
| 0.1 | 0.08474 | 0.00493 |
| 0.0 | 0.08437 | 0.00475 |

## Section VII: A new series expansion for displacement thickness

This section deals with new series expansions which have been developed for the flows $u_1 = u_0(\xi + \xi^3)$ and $u_1 = u_0(1 + \xi)$ from which the non-dimensional displacement thickness may be calculated for each problem.

In the course of calculating the parameters $\lambda$, $\mu$, H, l and L for the flows $u_1 = u_0(1 + \xi)$ and $u_1 = u_0(\xi + \xi^3)$ Blasius series expansions were found for the non-dimensional displacement thickness in each case. These were:-

for the flow $u_1 = u_o(1+\xi)$, where $I_1(\xi) = (\frac{u_o}{\nu c})^{\frac{1}{2}}\delta_1$

$$I_1(\xi) = 1.720770\xi^{\frac{1}{2}} - 4.476370\xi^{3/2} + 16.585170\xi^{5/2} - 74.594770\xi^{7/2} + 389.986770\xi^{9/2} - 2267.593170\xi^{11/2} + \ldots \qquad (7.1)$$

and for the flow $u_1 = u_o(\xi + \xi^3)$

$$I_1(\xi) = 0.647900 - 0.761796\xi^2 + 1.205206\xi^4 - 2.001326\xi^6 + 3.398026\xi^8 - 5.887926\xi^{10} + \ldots \qquad (7.2)$$

The usefulness of (7.1) and (7.2) is limited by the fact that for all but the smallest values of $\xi$, these series expansions do not converge (in the sense that the contributions from the last few terms become negligible).

A similar problem was encountered by Van Dyke[27] in his paper calculating second order boundary layer effects on a parabola in a uniform stream. For the purposes of his problem Van Dyke required the skin friction on the parabola and the Blasius series for this he found to be divergent (in the above sense). He proceeded to improve the 'convergence' of his series as follows. By non-dimensionalising his co-ordinate along the surface with respect to the nose radius he noted that far downstream on the parabola the skin friction approaches that for a flat plate, because the nose radius becomes negligible relative to the dimensions of interest. His Howarth-Blasius series for the non-dimensional skin friction he then proceeded to stretch by firstly recasting it in terms of the natural (parabolic) co-ordinate for the problem and then applying an Euler transform to an appropriately chosen function related to this series. His final assertion, that from this series the skin friction everywhere on the parabola may be determined, he justified by producing (from this series) at the limiting value of his independent variable the skin friction for the flat plate to within 1.3% of the computed value.

A further examination of (7.1) and (7.2), with the above work of Van Dyke in mind, reveals a possible method of obtaining more from these than can be gained from them in their present forms. As $\xi$ becomes large $u_1 = u_0(1 + \xi) \sim u_0\xi$ and $u_0(\xi + \xi^3) \sim u_0\xi^3$ and the corresponding non-dimensional displacement thicknesses should behave as $I_1 \sim C$ and $I_1 \sim A/\xi$ respectively, where A and C are known constants. The series expansions, $I_1$ for $u_1 = u_0(1 + \xi)$ and $\xi I_1$ for $u_1 = u_0(\xi + \xi^3)$, should therefore tend to constants as $\xi$ tends to infinity, and though this clearly does not happen with (7.1) and (7.2) it would provide a check on new series expansions were suitable co-ordinates to be found.

These co-ordinates may be found as follows. Both these problems have associated unfavourable pressure gradient flows; $u_1 = u_0(1 - \xi)$ which separates where $\xi = 0.120$; and $u_1 = u_0(\xi - \xi^3)$ which separates where $\xi = 0.655$. It is known that the displacement thickness has a square root singularity at separation (see Appendix) and since our series expansions for displacement thickness for both favourable and unfavourable pressure gradients are of the same form it is possible that this singularity is limiting the convergence of the favourable pressure gradient expansion. The nature and position of the singularity is confirmed by using Domb's ratio test (see Appendix). Under these circumstances it has been shown by Bellman[1] that an Euler transformation may be used to improve the convergence of a series expansion, and since the singularity in the displacement thickness is square root in nature we apply the transform to $I_1^2$ (for $u_1 = u_o(1+\xi)$) and to $(\xi I_1)^2$ for $u_1 = u_o(\xi + \xi^3)$.

For the problem $u_1 = u_o(1+\xi)$ we have

$$I_1^2 = 2.961049\xi - 15.405606\xi^2 + 77.116414\xi^3 - 405.203598\xi^4 + 2285.050512\xi^5 - 13769.798460\xi^6 + \ldots$$

and in terms of $y = \xi/(\xi + 0.120)$

$$I_1^2 = 0.355326y + 0.133485y^2 + 0.044901y^3 + 0.005551y^4 - 0.011736y^5 - 0.018396y^6 + \ldots$$

This yields the value $I_1^2 = 0.509131$ or $I_1 \approx 0.7135$ when $y = 1$. The computed value is $I_1 = 0.6479$.

For the problem $u_1 = u_o(\xi + \xi^3)$ we have

$$(\xi I_1)^2 = 0.419774\xi^2 - 0.987135\xi^4 + 2.142039\xi^6 - 4.429560\xi^8 + 8.904887\xi^{10} - 17.630800\xi^{12} + \ldots$$

and in terms of $y = \xi^2/(\xi^2 + 0.429025)$

$$(\xi I_1)^2 = 0.180093y - 0.001601y^2 - 0.014145y^3 - 0.007608y^4 - 0.002627y^5 - 0.000353y^6 + \ldots$$

This yields $(\xi I_1)^2 = 0.15395$ or $I_1 = 0.39237$ when $y = 1$. The computed value is $I_1 = 0.394482$.

It is interesting to note that the above two flows are special cases of the more general flow $u_1 = u_0(\xi^m + \xi^n)$. Recently Frossling[7] has developed new, general series expansions within the boundary layer which may be used to give the initial development of such flows. It would be interesting to examine these more general flows in the light of the above analysis to see whether there is a general principle underlying the above transformations.

# CHAPTER 3: Attempts at improving the tabulated functions $F_o(\lambda)$, $G_o(\lambda)$, $F_1(\lambda)$ and $G_1(\lambda)$

## Section I. Introductory Remarks

This chapter deals with the attempts which have been made to improve the method by making some alterations to the tabulated functions $F_o(\lambda)$, $G_o(\lambda)$, $F_1(\lambda)$ and $G_1(\lambda)$.

The next section deals with the collection of the data. It deals with the solutions which have been used to supply the parameter sets $\left\{\lambda, \mu, L, l^2, H\right\}$ and gives a brief outline of what was given tabulated and what had to be calculated.

The third section deals with the calculation of new forms for the functions $F_o(\lambda)$, $G_o(\lambda)$, $F_1(\lambda)$ and $G_1(\lambda)$. A large number of points are selected from the solutions given in the second section, to represent as wide a range of the parameter $\lambda$ as possible and also give as equal a representation to each solution as possible. The above functions are then represented by polynomial models in $\lambda$ and the coefficients in these determined by the method of least squares fitting using the selected points. For small values of $\lambda$ an additional analysis is carried out to ensure that the leading terms in these polynomial models are adjusted to produce a form for the shape parameter H which is continuous at $\lambda = 0$. Some of the more interesting details in the fitting procedure are mentioned and the functions which result are given as a numerical tabulation. This section, and the chapter, is concluded with some remarks on these new tables and certain relevant related quantities.

## Section II: Collection of Data

As described in the introduction to the chapter, this section deals with the data used for the construction of the method. For each solution a set of parameters $\left\{ \lambda, \mu, L, l^2, H \right\}$ was calculated and tabulated. These are described as follows.

3.2.1: The Falkner Skan solutions, $u_1 = u_o\xi^n$; $\xi = \frac{x}{c}$

3.2.2: The Howarth solution, $u_1 = u_o(1 - \xi)$; $\xi = \frac{x}{c}$

3.2.3: The Tani solution, $u_1 = u_o(1 - \xi^2)$; $\xi = \frac{x}{c}$

3.2.4: P. G. Williams' Solution

3.2.5: The flow, $u_1 = u_o(\xi + \xi^3)$

3.2.6: The flow; $u_1 = u_o(1 + \xi)$

3.2.7: Terrill's solution; $u_1 = u_o \sin\xi$

From each of these solutions some points were chosen to cover the whole range and give as equal a representation as possible to each solution.

### 3.2.1 The Falkner Skan solutions, $u_1 = u_o\xi^n$, $\xi = \frac{x}{c}$

On defining the stream function

$$\psi(\xi,\eta) = \left(\frac{2}{n+1}\right)^{\frac{1}{2}}(u_1 \nu x)^{\frac{1}{2}} f(\eta) = \left(\frac{2u_o}{n+1}\nu c\right)^{\frac{1}{2}}\left(\frac{x}{c}\right)^{\frac{n+1}{2}} f(\eta)$$

where

$$\eta = \left(\frac{n+1}{2}\right)^{\frac{1}{2}}\left(\frac{u_1}{\nu x}\right)^{\frac{1}{2}} y = \left\{\frac{(n+1)u_o}{2\nu c}\right\}^{\frac{1}{2}}\left(\frac{x}{c}\right)^{\frac{n-1}{2}} y$$

the function $f(\eta)$ satisfies the equation

$$f''' + ff'' + \beta(1 - f'^2) = 0 \text{ where } \beta = \frac{2n}{n+1}$$

and

$$f(0) = f'(0) = 0, \; f' \to 1 \text{ as } \eta \to \infty$$

$$\left(\frac{\partial u}{\partial y}\right)_w = \left(\frac{n+1}{2}\right)^{\frac{1}{2}} \left(\frac{u_1}{\nu x}\right)^{\frac{1}{2}} u_1 f''(0)$$

$$\delta_1 = \left(\frac{2}{n+1}\right)^{\frac{1}{2}}\left(\frac{\nu x}{u_1}\right)^{\frac{1}{2}} \int_0^\infty (1-f')d\eta = \left(\frac{2}{n+1}\right)^{\frac{1}{2}}\left(\frac{\nu x}{u_1}\right)^{\frac{1}{2}} \delta_1^* \text{ where } \delta_1^* = \int_0^\infty (1-f')d\eta$$

$$\delta_2 = \left(\frac{2}{n+1}\right)^{\frac{1}{2}}\left(\frac{\nu x}{u_1}\right)^{\frac{1}{2}} \int_0^\infty f'(1-f')d\eta = \left(\frac{2}{n+1}\right)^{\frac{1}{2}}\left(\frac{\nu x}{u_1}\right)^{\frac{1}{2}} \delta_2^* \text{ where } \delta_2^* = \int_0^\infty f'(1-f')d\eta$$

In terms of $\xi$ these become, $\delta_1 = \left(\frac{2}{n+1}\right)^{\frac{1}{2}}\left(\frac{\nu\xi}{u_1}\right)^{\frac{1}{2}} C^{\frac{1}{2}}\delta_1^*$; $\delta_2 = \left(\frac{2}{n+1}\right)^{\frac{1}{2}}\left(\frac{\nu\xi}{u_1}\right)^{\frac{1}{2}} C^{\frac{1}{2}}\delta_2^*$

$$\therefore \lambda = \frac{\delta_2^2 u_1^1}{\nu} = \frac{2}{n+1} \cdot \frac{\nu\xi}{u_\circ \xi^n} C \frac{(\delta_2^*)^2}{\nu} n.\ u_\circ \frac{\xi^{n-1}}{C} = \frac{2n}{n+1} (\delta_2^*)^2$$

$$l = \frac{\delta_2}{u_1} \left(\frac{\partial u}{\partial y}\right)_w = \left(\frac{2}{n+1}\right)^{\frac{1}{2}} \left(\frac{\nu\xi}{u_\circ\xi^n}\right)^{\frac{1}{2}} C^{\frac{1}{2}} \delta_2^* \left(\frac{n+1}{2}\right)^{\frac{1}{2}} \left(\frac{u_\circ\xi^n}{\nu\xi}\right)^{\frac{1}{2}} \frac{C^{-\frac{1}{2}} u_\circ \xi^n f''(0)}{u_\circ\xi^n}$$

$$= f''(0)\delta_2^*$$

$$H = \delta_1^*/\delta_2^*$$

$$\mu = \frac{u_1 u_1''}{(u_1')^2}\lambda^2 = \frac{u_\circ\xi^n\ n(n-1)u_\circ\xi^{n-2}}{n^2 u_\circ^2 \xi^{2n-2}/C^2} \cdot \frac{1}{C^2}$$

$$= \frac{(n-1)}{n}\lambda^2$$

Also $L = 2\left\{l - \lambda(H+2)\right\}$; but $f''(0) = \beta\,\delta_1^* + (1+\beta)\delta_2^*$

$$\therefore L = 2\left\{(\beta\delta_1^* + (1+\beta)\delta_2^*)\delta_2^* - \beta(\delta_2^*)^2 \left(\frac{\delta_1^*}{\delta_2^*} + 2\right)\right\} = 2\left\{(1+\beta)\delta_2^{*2} - 2\beta\delta_2^{*2}\right\}$$

$$= 2\left\{(1-\beta)\delta_2^{*2}\right\} = 2\left(\frac{1}{\beta} - 1\right)\lambda$$

But $\mu = -2\left(\frac{1}{\beta} - 1\right)\lambda^2 \quad \therefore \quad \frac{L}{\mu} = -\frac{1}{\lambda} \quad \therefore L = -\mu/\lambda$

An alternative form of the Falkner-Skan equation (useful for large values of $\beta$) in similar co-ordinates $(\zeta,\theta)$ is assuming no mass transfe

$$\theta''' + \frac{1}{\beta}\,\theta\,\theta'' + 1 - \theta'^2 = 0,\quad s = 0,\ \theta = \theta' = 0,\quad s \to \infty,\ \theta' \to 1,$$

where

$$s = y\left(\frac{1}{\gamma}\,\frac{du_1}{dx}\right)^{\frac{1}{2}}\ ;\quad \theta = \frac{\psi}{u_1}\left(\frac{1}{\gamma}\ \frac{du_1}{dx}\right)^{\frac{1}{2}}$$

Useful Relationships

$$\delta_1^{**} = \int_0^{\infty}\left(1 - \frac{d\theta}{ds}\right) ds;\quad \delta_2^{**} = \int_0^{\infty}\frac{d\theta}{ds}\left(1 - \frac{d\theta}{ds}\right) ds$$

and

$$\beta\,\delta_1^{**} + (1 + \beta)\,\delta_2^{**} = \beta\,\theta_0''$$

The two forms of the Falkner Skan equation are related by

$$\delta_1^{**} = \beta^{\frac{1}{2}}\,\delta_1^{*};\ \delta_2^{**} = \beta^{\frac{1}{2}}\,\delta_2^{*};\ f''(0) = \beta^{\frac{1}{2}}\theta_0''$$

In terms of these quantities

$$\lambda = (\delta_2^{**})^2\ ;\ l = \theta_0''\,\delta_2^{**};\ H = \frac{\delta_1^{**}}{\delta_2^{**}}$$

Results for Falkner-Skan Solution

A. Using Hartree's [8] values for $\beta = -0.19,\ -0.18,\ -0.16,\ -0.14$

the following table is obtained.

| $\beta$ | $\lambda$ | $\mu$ | L | $l^2$ | H |
|---|---|---|---|---|---|
| -0.19 | -0.063202 | 0.050030 | 0.791874 | 0.002460 | 3.47982 |
| -0.18 | -0.057994 | 0.044093 | 0.760230 | 0.005320 | 3.29668 |
| -0.16 | -0.048811 | 0.034539 | 0.707500 | 0.011071 | 3.09170 |
| -0.14 | -0.040643 | 0.026904 | 0.661545 | 0.016652 | 2.96348 |

B. Using the equation $f''' + ff'' + \beta(1 - f'^2) = 0$, $f(0) = f'(0) = 0$, $f' \to 1$ as $\eta \to \infty$, with values given in reference 6.

| $\beta$ | $\lambda$ | $\mu$ | L | $l^2$ | H |
|---|---|---|---|---|---|
| 0.1 | 0.0189619 | -0.006472 | 0.341323 | 0.065347 | 2.4809 |
| 0.2 | 0.0333299 | -0.008887 | 0.266643 | 0.078588 | 2.4108 |
| 0.3 | 0.0446389 | -0.009299 | 0.208310 | 0.089314 | 2.3617 |
| 0.4 | 0.0537849 | -0.008678 | 0.161353 | 0.098162 | 2.3252 |
| 0.5 | 0.0613446 | -0.007526 | 0.122689 | 0.105585 | 2.296938 |
| 0.6 | 0.067701 | -0.006111 | 0.090267 | 0.111898 | 2.2743 |
| 0.8 | 0.077800 | -0.003026 | 0.038896 | 0.122049 | 2.2404 |
| 1.0 | 0.085465 | 0.0 | 0.0 | 0.129844 | 2.2162295 |
| 1.2 | 0.091491 | 0.002790 | -0.030503 | 0.136028 | 2.1979 |
| 1.6 | 0.100336 | 0.007550 | -0.075257 | 0.145174 | 2.1724 |
| 2.0 | 0.106528 | 0.011348 | -0.106534 | 0.151627 | 2.1553 |
| -0.1 | -0.026527 | 0.015481 | +0.583594 | 0.027040 | 2.8011165 |
| -0.19883768 | -0.068148 | 0.056002 | 0.821763 | 0.0 | 4.0292280 |

C. Using the equation $\theta''' + \frac{1}{\beta}\theta\theta'' + 1 - \theta'^2 = 0$, $\xi = 0$, $\theta = \theta' = 0$; $\xi \to \infty, \theta' \to 1$, with values given in reference 6.

| $\frac{1}{\beta}$ | $\lambda$ | $\mu$ | L | $l^2$ | H |
|---|---|---|---|---|---|
| 1.3 | 0.076429 | -0.003505 | 0.045857 | 0.120661 | 2.24492 |
| 1.2 | 0.079219 | -0.002510 | 0.031687 | 0.123485 | 2.23588 |
| 1.1 | 0.082222 | -0.001352 | 0.016444 | 0.126537 | 2.22633 |
| 1.0 | 0.085465 | 0.0 | 0.0 | 0.129844 | 2.21623 |
| 0.9 | 0.088977 | 0.001583 | -0.017795 | 0.133441 | 2.20552 |
| 0.8 | 0.092793 | 0.003444 | -0.037117 | 0.137366 | 2.19416 |
| 0.7 | 0.096955 | 0.005640 | -0.058173 | 0.141666 | 2.18207 |
| 0.6 | 0.101512 | 0.008244 | -0.081209 | 0.146394 | 2.16918 |
| 0.5 | 0.106522 | 0.011347 | -0.106522 | 0.151618 | 2.15541 |
| 0.4 | 0.112057 | 0.015068 | -0.134469 | 0.157418 | 2.14067 |
| 0.3 | 0.118204 | 0.019561 | -0.165485 | 0.163888 | 2.12485 |
| 0.2 | 0.125067 | 0.025027 | -0.200108 | 0.171149 | 2.10784 |
| 0.1 | 0.132777 | 0.031734 | -0.238999 | 0.179347 | 2.08950 |
| 0.0 | 0.141497 | 0.040043 | -0.282995 | 0.188663 | 2.06969 |

It is interesting to note that, for the Falkner Skan values a useful check on the accuracy of L may be carried by calculating $L = -\mu/\lambda$ and comparing it with $L = 2\left\{1 - \lambda(H + 2)\right\}$. This has been done in the table which follows. From this we may say that we can rely on the values for L in group A to at least three figures, in group B to at least four figures and in group C to at least five figures.

The Falkner-Skan values

| A. $\beta$ | $\lambda$ | $\mu$ | $-\mu/\lambda$ | L |
|---|---|---|---|---|
| -0.19 | -0.063202 | 0.050030 | 0.791589 | 0.791874 |
| -0.18 | -0.057994 | 0.044093 | 0.760303 | 0.760230 |
| -0.16 | -0.048811 | 0.034539 | 0.707609 | 0.707500 |
| -0.14 | -0.040643 | 0.026904 | 0.661959 | 0.661545 |
| B. | | | | |
| 0.1 | 0.0189619 | -0.006472 | 0.341316 | 0.341323 |
| 0.2 | 0.0333299 | -0.008887 | 0.266637 | 0.266643 |
| 0.3 | 0.0446389 | -0.009299 | 0.208316 | 0.208310 |
| 0.4 | 0.0537849 | -0.008678 | 0.161346 | 0.161353 |
| 0.5 | 0.0613446 | -0.007526 | 0.122684 | 0.122689 |
| 0.6 | 0.067701 | -0.006111 | 0.090264 | 0.090267 |
| 0.8 | 0.077800 | -0.003026 | 0.038895 | 0.038896 |
| 1.0 | 0.085465 | 0.0 | 0.0 | 0.0 |
| 1.2 | 0.091491 | 0.002790 | -0.030495 | -0.030503 |
| 1.6 | 0.100336 | 0.007550 | -0.075247 | -0.075257 |
| 2.0 | 0.106528 | 0.011348 | -0.106526 | -0.106534 |
| -0.1 | -0.026527 | 0.015481 | 0.583594 | 0.583594 |
| -0.19883768 | -0.068148 | 0.056002 | 0.821770 | 0.821763 |
| C. | | | | |
| $1/\beta$ | | | | |
| 1.3 | 0.076429 | -0.003505 | 0.045859 | 0.045857 |
| 1.2 | 0.079219 | -0.002510 | 0.031684 | 0.031687 |
| 1.1 | 0.082222 | -0.001352 | 0.016443 | 0.016444 |
| 1.0 | 0.085465 | 0.0 | 0.0 | 0.0 |
| 0.9 | 0.088977 | 0.001583 | -0.017791 | -0.017795 |
| 0.8 | 0.092793 | 0.003444 | -0.037115 | -0.037117 |
| 0.7 | 0.096955 | 0.005640 | -0.058171 | -0.058173 |
| 0.6 | 0.101512 | 0.008244 | -0.081212 | -0.081209 |
| 0.5 | 0.106522 | 0.011347 | -0.106522(5) | -0.106522 |
| 0.4 | 0.112057 | 0.015068 | -0.134467 | -0.134469 |
| 0.3 | 0.118204 | 0.019561 | -0.165485 | -0.165485 |
| 0.2 | 0.125067 | 0.025027 | -0.200109 | -0.200108 |
| 0.1 | 0.132777 | 0.031734 | -0.239002 | -0.238999 |
| 0.0 | 0.141497 | 0.040043 | -0.282995 | -0.282995 |

3.2.2 <u>The Howarth Solution, $u_1 = u_o(1-\xi)$, $\xi = \frac{x}{c}$</u>

In Howarth's[10] notation, the flow is $U = b_o - b_1 x = b_o(1 - X^*)$

where $X^* = \frac{b_1}{b_o}\, x$

$(\text{i.e. } U \equiv u_1,\ b_o = u_o,\ \frac{b_1}{b_o} \equiv \frac{1}{c})$

Howarth solves this problem by series expansion techniques and on p.561 (10), the following quantities are tabulated.

$$\text{Col } 1 \equiv X^*,\ \text{Col } 2 \equiv \frac{\nu^{\frac{1}{2}}(\frac{\partial u}{\partial y})_o}{b_o b_1^{\frac{1}{2}}};\ \text{Col } 3 \equiv \frac{b_1^{\frac{1}{2}}\delta_1}{\frac{1}{2}},\ \text{Col } 4 \equiv \frac{b_1^{\frac{1}{2}}\delta_2}{\nu^{\frac{1}{2}}}$$

Let Col 2 = $C_2$, Col 3 = $C_3$, Col 4 = $C_4$

$$\therefore \delta_2 = \frac{\nu^{\frac{1}{2}}C_4}{b_1^{\frac{1}{2}}};\ \delta_1 = \frac{\nu^{\frac{1}{2}}C_3}{b_1^{\frac{1}{2}}};\ (\frac{\partial u}{\partial y})_o = b_o b_1^{\frac{1}{2}} C_2 \nu^{-\frac{1}{2}}$$

$$\therefore \lambda = \frac{\delta_2^2}{\nu} u_1' = -\frac{\nu}{b_1} C_4^2 \frac{b_1}{\nu} = -C_4^2$$

$$l = \frac{\delta_2}{u_1}(\frac{\partial u}{\partial y})_o = \frac{\nu^{\frac{1}{2}}C_4}{b_1^{\frac{1}{2}} b_o(1 - X^*)}\, b_o b_1^{\frac{1}{2}} C_2 \nu^{-\frac{1}{2}} = \frac{C_2 C_4}{1 - X^*}$$

$$H = \frac{\delta_1}{\delta_2} = \text{Col } 3/\text{Col } 4 = C_3/C_4,\ \mu \equiv 0.$$

| $X^*$ | $\lambda$ | L | $l^2$ | H |
|---|---|---|---|---|
| 0.0000 | 0.000 | 0.441056 | 0.04863 | 2.591053 |
| 0.0125 | -0.00578 | 0.474889 | 0.04441(5) | 2.6184 |
| 0.0250 | -0.01210 | 0.512379 | 0.03995 | 2.6545 |
| 0.0375 | -0.01877 | 0.549378 | 0.03471(5) | 2.7080 |
| 0.0500 | -0.02624 | 0.594503 | 0.02971 | 2.7592 |
| 0.0625 | -0.03460 | 0.646576 | 0.02459 | 2.8118 |
| 0.0750 | -0.04368 | 0.703911 | 0.01920 | 2.8852 |
| 0.0875 | -0.05336 | 0.765051 | 0.01350 | 2.9913 |
| 0.1000 | -0.06452 | 0.836718 | 0.00768 | 3.1260 |
| 0.1125 | -0.07618 | 0.920220 | 0.00258 | 3.3732 |
| 0.1200 | -0.08410 | 0.980202 | 0.00000 | 3.8276 |

3.2.3 The Tani[21] solution. $u_1 = u_o(1 - \xi^2),\ \xi = \frac{x}{c}$

In Tani's notation the flow at the edge of the boundary layer is given as $u_1 = u_o + \partial x^n$. Tani solves this for n = 2, n = 6, n = 8. The velocity distribution outside this boundary layer is of the form $u_1 = u_o + \partial x^n$, where x is the distance measured along the surface and $\partial$, n and $u_o$ are constants. Assume an expression of the form $\phi = \sqrt{\nu x u_o} \sum_{r=0}^{\infty} \xi_1^r f_r(\eta)$, for the stream function $\phi$, where $\xi_1 = \frac{8\partial x^n}{u_o}$,

$\eta = \frac{y}{2}\sqrt{\frac{u_o}{\nu x}}$, and y is the distance measured normal to the surface. Substitution of these expressions into the boundary layer equation and equating coefficients of power of $\xi_1$, yields differential equations for the $f_r$.

For the flow $u_1 = u_o + \partial x^2$

Denote $\delta_1^* = \int_0^\infty (1 - \frac{u}{u_1})\, d\eta$; $\delta_2^* = \int_0^\infty \frac{u}{u_1}(1 - \frac{u}{u_1})\, d\eta$

$$\therefore \delta_1 = 2\sqrt{\frac{\nu x}{u_o}} \int_0^\infty (1 - \frac{u}{u_1})\, d\eta = 2\sqrt{\frac{\nu x}{u_o}}\, \delta_1^*$$

For $u_1 = u_o + \partial x^2$

$$\xi_1 = \frac{8\partial x^2}{u_o}$$

$$\delta_2 = 2\sqrt{\frac{\nu x}{u_o}} \int_0^\infty \frac{u}{u_1}(1 - \frac{u}{u_1})\, d\eta = 2\sqrt{\frac{\nu x}{u_o}}\, \delta_2^*$$

$$\therefore H = \frac{\delta_1}{\delta_2} = \frac{\delta_1^*}{\delta_2^*}$$

$$\therefore \lambda = \frac{\delta_2^2 u_1'}{\nu} = \frac{4\nu x}{u_o}\frac{(\delta_2^*)^2}{\nu}\, 2\partial x = \frac{8}{u_o}(\delta_2^*)^2\, \partial x^2 = \xi_1 (\delta_2^*)^2$$

$$\mu = \frac{\lambda^2 u_1 u_1''}{(u_1')^2} = \frac{\lambda^2 (u_o + \partial x^2) 2\partial}{4\partial^2 x^2} = \frac{\lambda^2 (u_o + u_o \xi_1/8) . 2\partial}{\partial u_o \xi_1 / 2}$$

$$= \frac{4\lambda^2 (1 + \xi_1/8)}{\xi_1}$$

For positive values of $\xi_1$, Tani obtained his results by means of a series expansion the accuracy of which cannot be estimated since the coefficients in it have not been quoted.

Results (Tani's solution)

| $\xi_1$ | $\lambda$ | $\mu$ | L | $l^2$ | H |
|---|---|---|---|---|---|
| 0.60 | 0.0492 | 0.0173 | 0.160319 | 0.087616 | 2.387 |
| 0.30 | 0.0284 | 0.0111 | 0.276331 | 0.070225 | 2.466 |
| 0.0 | 0.0 | 0.0 | 0.4410 | 0.048620 | 2.591 |
| -0.10 | -0.0116 | -0.0054 | 0.511057 | 0.040642 | 2.649 |
| -0.20 | -0.0247 | -0.0119 | 0.592417 | 0.032256 | 2.721 |
| -0.30 | -0.0392 | -0.0197 | 0.684618 | 0.023593 | 2.814 |
| -0.40 | -0.0557 | -0.0295 | 0.794027 | 0.014835 | 2.941 |
| -0.50 | -0.0742 | -0.0412 | 0.922615 | 0.006304 | 3.147 |
| -0.56 | -0.0864 | -0.0496 | 1.013474 | 0.001706 | 3.387 |
| -0.586 | -0.0916 | -0.0531 | 1.055232 | 0.0 | 3.760 |

### 3.2.4 P. G. Williams solution

The parameters tabulated below are for the incompressible flow, associated through the Stewartson-Illingworth transformation with the compressible boundary layer problem with external velocity $u_1 = u_o(1-\xi)$, $\xi = x/c$, with zero heat transfer at the wall, Prandtl number equal to unity, viscosity proportional to the absolute temperature and Mach number at the leading edge equal to 4. The results from which the parameters have been calculated have been very kindly supplied to me by P. G. Williams of University College, London and the method which he used to solve this problem is contained in (20).

Remarks: In the usual notation the two dimensional, incompressible laminar boundary layer equation is $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_1 \frac{du_1}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$ and

$$\delta_1 = \int_0^\infty (1 - \frac{u}{u_1})\, dy;\ \delta_2 = \int_0^\infty \frac{u}{u_1} (1 - \frac{u}{u_1})\, dy;\ \tau_w = \mu (\frac{\partial u}{\partial y})_w$$

If we define $u' = \frac{u}{u_\circ}$, $v' = \frac{v}{u_\circ} R^{\frac{1}{2}}$, $x' = \frac{x}{c}$, $y' = R^{\frac{1}{2}} \frac{y}{c}$, $u_1^* = \frac{u_1}{u_\circ}$ where R is taken to be $R = \frac{u_\circ c}{\nu}$, the above equation becomes

$$u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} = u_1^* \frac{du_1^*}{dx'} + \frac{\partial^2 u'}{\partial y'^2}$$

For his solution P. G. Williams tabulates the following quantities.

$$X';\ \sqrt{\frac{x'}{u_1^*}} (\frac{\partial u'}{\partial y'})_w\ ;\ \frac{1}{\sqrt{x'}} \int_0^\infty (1 - \frac{u'}{u_1^*})\, dy';\ \frac{1}{\sqrt{x'}} \int_0^\infty \frac{u'}{u_1} (1 - \frac{u'}{u_1^*})\, dy'$$

But $\sqrt{\frac{x'}{u_1^*}}(\frac{\partial u'}{\partial y'})_w = \sqrt{\frac{x'}{u_1^*}} (\frac{\partial u}{\partial y})_w cR^{-\frac{1}{2}} = \sqrt{\frac{x'}{u_1^*}} (\frac{\nu c}{u_\circ^3})^{\frac{1}{2}} (\frac{\partial u}{\partial y})_w = \sqrt{\frac{x'}{u_1^*}}\ T = \text{Col. } 4$

$$\frac{1}{\sqrt{x'}}\int_0^\infty \frac{u'}{u_1^*} (1 - \frac{u'}{u_1^*})\, dy' = \frac{1}{\sqrt{x'}} \int_0^\infty \frac{u}{u_1} (1 - \frac{u}{u_1}) R^{\frac{1}{2}} \frac{dy}{c} = \frac{1}{\sqrt{x'}} (\frac{u_\circ}{\nu c})^{\frac{1}{2}} \delta_2 = \text{Col. } 6$$

$$\frac{1}{\sqrt{x'}} \int_0^\infty (1 - \frac{u'}{u_1^*})\, dy' = \frac{1}{\sqrt{x'}} \int_0^\infty (1 - \frac{u}{u_1}) R^{\frac{1}{2}} \frac{dy}{c} = \frac{1}{\sqrt{x'}} (\frac{u_\circ}{\nu c})^{\frac{1}{2}} \delta_1 = \text{Col. } 5$$

$$\lambda = \frac{\delta_2^2}{\nu} \frac{du_1}{dx} = \frac{(\text{Col } 6)^2}{\nu} x' (\frac{\nu c}{u_\circ}) \frac{u_\circ}{c} \frac{du_1^*}{dx'} = x' (\text{Col } 6)^2 \frac{du_1}{dx'}$$

$$l = \frac{\delta_2}{u_1} (\frac{\partial u}{\partial y})_w = (\text{Col } 6) \sqrt{x'} (\frac{\nu c}{u_\circ})^{\frac{1}{2}} \cdot \frac{1}{u_1} (\text{Col } 4) \frac{u_1^*}{x'} (\frac{u_\circ^3}{\nu c})^{\frac{1}{2}}$$

$$= (\text{Col } 6) \sqrt{x'} (\frac{\nu c}{u_\circ})^{\frac{1}{2}} \frac{1}{u_1^* u_\circ} (\text{Col } 4)\, u_1^* (\frac{u_\circ^3}{\nu c})^{\frac{1}{2}}$$

$$= \text{Col } 6 \times \text{Col } 4$$

$$H = \frac{\delta_1}{\delta_2} = \text{Col } 5/\text{Col } 6$$

$$\mu = \frac{\lambda^2 u_1 u_1''}{(u_1')^2} = \lambda^2 \frac{u_1^*(u_1^*)''}{((u_1^*)')^2}$$

Differentiation wrt x          Differentiation wrt x'

Results from P. G. Williams' Solution

| $\lambda$ | $\mu$ | L | $l^2$ | H |
|---|---|---|---|---|
| 0.0 | 0.0 | 0.44096 | 0.048611 | 2.5916 |
| -0.01316 | 0.00143 | 0.51623 | 0.03867 | 2.6697 |
| -0.02392 | 0.00466 | 0.57847 | 0.03087 | 2.7454 |
| -0.03290 | 0.00870 | 0.63069 | 0.02458 | 2.8200 |
| -0.04048 | 0.01302 | 0.67542 | 0.01948 | 2.8940 |
| -0.04696 | 0.01734 | 0.71392 | 0.01529 | 2.9682 |
| -0.05250 | 0.02148 | 0.74746 | 0.01187 | 3.0429 |
| -0.05726 | 0.02533 | 0.77658 | 0.00906 | 3.1186 |
| -0.06133 | 0.02885 | 0.80199 | 0.00678 | 3.1954 |
| -0.06479 | 0.03199 | 0.82388 | 0.00493 | 3.2736 |
| -0.06770 | 0.03474 | 0.84271 | 0.00347 | 3.3534 |
| -0.06872 | 0.03572 | 0.84939 | 0.00298 | 3.3858 |
| -0.07010 | 0.03708 | 0.85852 | 0.00233 | 3.4349 |
| -0.07132 | 0.03828 | 0.86673 | 0.00178 | 3.4845 |
| -0.07237 | 0.03934 | 0.87390 | 0.00132 | 3.5347 |
| -0.07327 | 0.04026 | 0.88019 | 0.00095 | 3.5853 |
| -0.07403 | 0.04104 | 0.88549 | 0.00065 | 3.6363 |
| -0.07464 | 0.04167 | 0.88995 | 0.00042 | 3.6876 |
| -0.07513 | 0.04217 | 0.89347 | 0.00024 | 3.7390 |
| -0.07548 | 0.04255 | 0.89620 | 0.00012 | 3.7901 |
| -0.07572 | 0.04280 | 0.89802 | 0.00004 | 3.8407 |
| -0.07585 | 0.04293 | 0.89908 | 0.00000(8) | 3.8898 |

Section 3.2.5 The flow; $u_1 = u_0 \, (\xi + \xi^3)$

As this solution has been described in some detail in Chapter 2, only the relevant results will be quoted here.

| $\lambda$ | $\mu$ | L | $l^2$ | H |
|---|---|---|---|---|
| 0.086 | 0.000330 | -0.002811 | 0.130365 | 2.21473 |
| 0.087 | 0.000940 | -0.008050 | 0.131342 | 2.21192 |
| 0.088 | 0.001539 | -0.013271 | 0.132326 | 2.20911 |
| 0.089 | 0.002162 | -0.018475 | 0.133314 | 2.20629 |
| 0.090 | 0.002700 | -0.023658 | 0.134310 | 2.20347 |
| 0.091 | 0.003275 | -0.028818 | 0.135311 | 2.20062 |
| 0.092 | 0.003818 | -0.033957 | 0.136320 | 2.19776 |
| 0.093 | 0.004359 | -0.039071 | 0.137335 | 2.19488 |
| 0.094 | 0.004880 | -0.044157 | 0.138358 | 2.19196 |
| 0.095 | 0.005388 | -0.049233 | 0.139381 | 2.18900 |
| 0.096 | 0.005874 | -0.054133 | 0.140464 | 2.18596 |
| 0.097 | 0.006334 | -0.059003 | 0.141550 | 2.18282 |
| 0.098 | 0.006768 | -0.063796 | 0.142657 | 2.17958 |

Section 3.2.6 The flow $u_1 = u_0 \, (1+\xi)$

For this flow techniques were used which were similar to those used for the flow $u_1 = u_0 \, (\xi + \xi^3)$. The 'Blasius-Howarth' series expansion were found for the non-dimensional skin friction T, displacement thickness $I_1$ and momentum thickness $I_2$. From these series expansions the parameters $\lambda$, l, H were calculated in terms of $\xi$. The co-ordinate $\xi$ was then eliminated between l and $\lambda$, and H and $\lambda$ to give $l = l(\lambda)$, and $H = H(\lambda)$. Four, five and six term expansions in the parameter $\lambda$ were examined and adapted, as described in Chapter 2, to give a form which gave the correct limiting value corresponding to $\xi \to \infty$ ie $\lambda = 0.0855$.

| $\lambda$ | $\mu$ | L | $l^2$ | H |
|---|---|---|---|---|
| 0.000 | 0.0 | 0.441056 | 0.048326 | 2.59103 |
| 0.005 | 0.0 | 0.412636 | 0.052505 | 2.56447 |
| 0.010 | 0.0 | 0.384513 | 0.056475 | 2.53887 |
| 0.015 | 0.0 | 0.356682 | 0.060542 | 2.51415 |
| 0.020 | 0.0 | 0.329144 | 0.064707 | 2.49023 |
| 0.025 | 0.0 | 0.301897 | 0.068971 | 2.46702 |
| 0.030 | 0.0 | 0.274946 | 0.073336 | 2.44444 |
| 0.035 | 0.0 | 0.248295 | 0.077804 | 2.42244 |
| 0.040 | 0.0 | 0.221952 | 0.082377 | 2.40094 |
| 0.045 | 0.0 | 0.195928 | 0.087059 | 2.37987 |
| 0.050 | 0.0 | 0.170237 | 0.091856 | 2.35917 |
| 0.055 | 0.0 | 0.144899 | 0.096772 | 2.33877 |
| 0.060 | 0.0 | 0.119938 | 0.101815 | 2.31860 |
| 0.065 | 0.0 | 0.095383 | 0.106993 | 2.29860 |
| 0.070 | 0.0 | 0.071270 | 0.112317 | 2.27860 |
| 0.075 | 0.0 | 0.047643 | 0.117798 | 2.25861 |
| 0.080 | 0.0 | 0.024552 | 0.123450 | 2.23849 |
| 0.085 | 0.0 | 0.002059 | 0.129291 | 2.21813 |
| 0.085465 | 0.0 | 0.000000 | 0.129845 | 2.21622(6) |

### 3.2.7 Terrill's solution; $u_1 = u_\circ \sin\xi$

(Terrill uses $\chi'$ and this will be used below.)

Remarks: Terrill[22] uses the Görtler transformation which gives the velocity potential of the outer flow as

$$\phi = \int_0^{\chi} U(\chi')\, d\chi' \text{ where } \chi = 0 \text{ is the leading edge of}$$

the surface.

The independent variables are taken as

$$\xi = \frac{\phi}{u_\circ l^*}\,;\ \eta = \frac{Uy}{(2\nu\phi)^{\frac{1}{2}}} \text{ where } U_\circ \text{ and } l^* \text{ are a suitable reference}$$

velocity and length respectively.

If the mainstream velocity $U(\chi)$ is given by

$$U(\chi) = U_\circ f(\chi')$$

then the relations between the co-ordinates $(\xi,\eta)$ of Görtler's equation and the non-dimensional co-ordinates $(\chi', y')$ usually associated with the boundary-layer equation are $(\chi' = \frac{\chi}{l^*}\ y' = \frac{y}{\delta})$

$$\xi = \int_0^{\chi'} f(\chi')d\chi' = g(\chi');\ \eta = \left\{\frac{f(\chi')}{[2g(\chi')]^{\frac{1}{2}}}\right\} y'$$

$\therefore$ For the flow $U = U_\circ \sin\chi'$ we have

$$\phi = U_\circ \int_0^{\chi} \sin\frac{\theta}{l^*}\, d\theta = l^* u_\circ(1 - \cos\chi')$$

$$\xi = 1 - \cos\chi'$$

$$\therefore\ \delta_1 = \int_0^{\infty}\left(1 - \frac{u}{U}\right) dy = \frac{(2\phi\nu)^{\frac{1}{2}}}{U}\int_0^{\infty}\left(1 - \frac{\partial f}{\partial\eta}\right) d\eta$$

$$= \frac{(2l^* u_\circ \nu(1 - \cos\chi'))^{\frac{1}{2}}}{U_\circ \sin\chi'}\int_0\left(1 - \frac{\partial f}{\partial\eta}\right)d\eta$$

$$= \left(\frac{\nu l^*}{u_\circ}\right)^{\frac{1}{2}} \sec\frac{\chi'}{2}\int_0^{\infty}\left(1 - \frac{\partial f}{\partial\eta}\right)d\eta$$

Similarly $\delta_2 = \left(\frac{\nu l^*}{u_\circ}\right)^{\frac{1}{2}} \sec\frac{\chi'}{2}\int_0^{\infty} f'(1 - f')\, d\eta$

$$\left(\frac{\partial u}{\partial y}\right)_w = \frac{\rho}{\mu} U^2 \left(\frac{\nu}{2\phi}\right)^{\frac{1}{2}} f''(0)$$

$$= \frac{1}{\nu} U_\circ^{\,2} \sin^2\chi' \left(\frac{\nu}{2l^{*}.2\sin^2\frac{\chi'}{2}\; u_\circ}\right)^{\frac{1}{2}} f''(0) = \left(\frac{U_\circ^{\,3}}{\nu l^{*}}\right)\cdot \left(\sin\frac{\chi'}{2} . 2\cos^2\frac{\chi'}{2}\right) f''(0)$$

What is given tabulated by Terrill[22] is

$$\left(\frac{u_\circ}{\nu l^{*}}\right)^{\frac{1}{2}} \delta_1 ;\; \left(\frac{u_\circ}{\nu l^{*}}\right)^{\frac{1}{2}} \delta_2 ;\; \left(\frac{\nu l^{*}}{U_\circ^3}\right)^{\frac{1}{2}} \left(\frac{\partial u}{\partial y}\right)_w = I_1,\ I_2,\ T \text{ (respectively)}$$

$$\therefore\ l = \frac{\delta_2}{U}\left(\frac{\partial u}{\partial y}\right)_w = \left(\frac{\nu l^{*}}{U_\circ}\right)^{\frac{1}{2}} \frac{I_2}{U_\circ \sin\chi'} \left(\frac{U_\circ^3}{\nu l^{*}}\right)^{\frac{1}{2}} T$$

$$= \frac{I_2 T}{\sin\chi'}$$

$$H = \frac{\delta_1}{\delta_2} = I_1/I_2\,, \qquad L = 2\left\{1 - \lambda(H+2)\right\},$$

$$\mu = \lambda^2 \frac{UU''}{(U')^2} = -\lambda^2 \frac{(\sin\chi')^2}{(\cos\chi')^2} = -\lambda^2 (\tan\chi')^2$$

3.?.7 Results

| $\lambda$ | $\mu$ | L | $1^2$ | H |
|---|---|---|---|---|
| 0.085158 | -0.000189 | 0.001473 | 0.129490 | 2.217 |
| 0.082227 | -0.002018 | 0.016914 | 0.126699 | 2.226 |
| 0.076009 | -0.006125 | 0.050250 | 0.120848 | 2.243 |
| 0.068476 | -0.011373 | 0.091309 | 0.113951 | 2.263 |
| 0.055880 | -0.020659 | 0.160784 | 0.102834 | 2.300 |
| 0.046493 | -0.028047 | 0.213769 | 0.094900 | 2.327 |
| 0.035320 | -0.037336 | 0.277916 | 0.085842 | 2.361 |
| 0.025882 | -0.045523 | 0.332969 | 0.078459 | 2.390 |
| 0.016475 | -0.053972 | 0.388457 | 0.071323 | 2.421 |
| -0.008143 | -0.077713 | 0.537908 | 0.053941 | 2.507 |
| -0.017497 | -0.087143 | 0.596018 | 0.047743 | 2.544 |
| -0.035808 | -0.106668 | 0.712631 | 0.036408 | 2.622 |
| -0.062797 | -0.137259 | 0.891044 | 0.021457 | 2.762 |
| -0.074505 | -0.151379 | 0.971718 | 0.015744 | 2.837 |
| -0.091366 | -0.172255 | 1.091726 | 0.008338 | 2.975 |
| -0.099811 | -0.183026 | 1.154272 | 0.005097 | 3.067 |
| -0.106238 | -0.191430 | 1.203528 | 0.002893 | 3.158 |
| -0.111425 | -0.198271 | 1.244467 | 0.001314 | 3.259 |
| -0.115267 | -0.203435 | 1.276556 | 0.000325 | 3.381 |
| -0.116431 | -0.205077 | 1.286746 | 0.000078 | 3.450 |
| -0.116843 | -0.205621 | 1.29046 | 0.0 | 3.521 |

## Section III: The Fitting of the Forms

### III.1 Introduction

This section deals with methods for calculating forms for the functions $F_o(\lambda)$, $G_o(\lambda)$, $F_1(\lambda)$ and $G_1(\lambda)$ so that the approximations for L and $1^2$

$$L(\lambda,\mu) = F_o(\lambda) - \mu G_o(\lambda)$$
$$1^2(\lambda,\mu) = F_1(\lambda) - \mu G_1(\lambda)$$

yield values in as good agreement as possible, over the whole range of $\lambda$, with the data given in the previous section. Added to this we also require that the tabulated functions $F_o(\lambda)$, $G_o(\lambda)$, $F_1(\lambda)$ and $G_1(\lambda)$

are carefully enough defined for small $\lambda$ to give a form for H which is continuous at $\lambda = 0$. These aims were attempted as follows.

First of all it was decided to represent $F_o(\lambda)$, $G_o(\lambda)$, $F_1(\lambda)$ and $G_1(\lambda)$ as polynomials in $\lambda$, so that least squares fitting could be carried out on the data. As it would have been uneconomical to use all of the data calculated in the previous section, three tables 3.1, 3.2 and 3.3 were produced as sample data points over which the forms could be fitted and tested. The first table 3.1 consists of sixty two points which cover a range $-0.117 \leqslant \lambda \leqslant 0.14$ and give as equal a representation as possible to the various solutions for which the parameters were calculated. The second table 3.2 consists of sixteen points covering a range $-0.02 \leqslant \lambda \leqslant 0.02$ which could be used to carry out a separate analysis for small values of $\lambda$. At each point in the tables 3.1 and 3.2 the residual was calculated between the given value of L or $l^2$ and its least squares estimate. To ensure that these residuals were typical for all the data, the residual was also calculated at each point in the Table 3.3 which consists of twenty five points not used in the fitting. Any major difference in the size of residuals at fitted and unfitted points could mean that a polynomial model had become too complicated and had started to 'wiggle' through the data, reproducing the 'fitted' data extremely well but being unreliable elsewhere.

In treating the data it was possible to produce the functions $F_o(\lambda)$, $G_o(\lambda)$, $F_1(\lambda)$ and $G_1(\lambda)$ either by taking simple polynomial models over smaller overlapping ranges and then smoothing these curves into each other to produce the final forms or by taking more complicated models over the whole range. Initial attempts with the former approach with $F_o(\lambda)$, $F_1(\lambda)$ quadratic and $G_o(\lambda)$, $G_1(\lambda)$ linear proved rather inflexible and the second approach was preferred. It was found that a convenient way of analysing complicated models quickly was to use an IBM routine called REGRE which is used in the theory of multiple linear regression. A guide as to the orders of polynomial to be expected for $F_o(\lambda)$, and $F_1(\lambda)$ was obtained from an examination of the Howarth solution for $u_1 = u_o(1 - \xi)$ together with three points from the flow $u_1 = u_o(1 + \xi)$, for both of which solutions $\mu = 0$. Once the orders of $F_o(\lambda)$ and $F_1(\lambda)$ had been decided various orders of polynomial were taken for $G_o(\lambda)$ and $G_1(\lambda)$ and models compared by calculating the total sum of squared residuals, mean modulus residual and root mean square residual in each case. A similar analysis was carried out for small values of $\lambda$ using the sixteen points in Table 3.2. For this region the leading coefficients in the forms are also related to produce a form for H which is continuous at $\lambda = 0$ and these conditions on the coefficients are worked out.

It was found that the forms over the whole range and over the inner range were in good agreement over ranges $0.025 \geqslant |\lambda| \geqslant 0.015$. The final numerically tabulated functions were produced using the whole range functions for $|\lambda| \geqslant 0.025$, the inner range forms for $|\lambda| \leqslant 0.015$, the values in the intermediate range $0.015 \leqslant |\lambda| \leqslant 0.025$ being produced by interpolation between the forms. For $|\lambda| \leqslant 0.015$ the values of

H are given by a polynomial form in $\lambda$ and $\mu$.

The tables produced by these methods are given at the end of this section and relevant results are discussed. An outline of some of the more important steps in the fitting procedure is now given below.

## III.2 The Computer Program used in Fitting the Data for L and $l^2$

The models we wish to examine are of the form

$$L(\lambda,\mu) = \sum_{r=0}^{n} a_r\lambda^r + \mu \sum_{r=0}^{p} b_r\lambda^r$$

and
$$l^2(\lambda,\mu) = \sum_{r=0}^{m} c_r\lambda^r + \mu \sum_{r=0}^{q} d_r\lambda^r$$

where n, p, m and q are positive integers and, where, to give agreement with the original forms for L and $l^2$

$$L(\lambda,\mu) = F_o(\lambda) - \mu G_o(\lambda)$$

$$l^2(\lambda,\mu) = F_1(\lambda) - \mu G_1(\lambda)$$

we take

$$F_o(\lambda) = \sum_{r=0}^{n} a_r\lambda^r; \quad G_o(\lambda) = -\sum_{r=0}^{p} b_r\lambda^r$$

$$F_1(\lambda) = \sum_{r=0}^{m} c_r\lambda^r; \quad G_1(\lambda) = -\sum_{r=0}^{q} d_r\lambda^r$$

The problems with which we are faced in fitting forms to our data are similar to those occurring in multiple linear regression. Given a set of observations of the variables Y, $X_1$, ..., $X_k$, say, where Y is thought to be a function of the others, multiple linear regression is concerned with the problem of estimating the variable Y by means of a linear function of the remaining variables. If the variable used to estimate Y is denoted by Y', the linear estimating functions may be expressed as

$$Y' = C_o + C_1X_1 + \dots\dots\dots C_kX_k$$

where the C's, the regression coefficients, are to be determined by means of available data. As in the case of two variables the unknown coefficients are estimated by the method of least squares. The derivation of the equations of the least squares fitting places no restriction on the nature of the variables $X_1, X_2, \dots X_k$ which may therefore be related in any manner desired. With this in mind it can be seen that we can easily adapt our models for L and $l^2$ into forms to which the programs for multiple linear regression may be applied.

By defining $X_r = \lambda^r$, $Y_r = \mu\lambda^r = \mu X_r$; $r = 0, 1, \dots$ the models then become linear in $X_r$ and $Y_r$

$$\text{i.e.} \quad L = a_o + \sum_{r=1}^{n} a_rX_r + \sum_{r=0}^{p} b_rY_r$$

$$l^2 = C_o + \sum_{r=1}^{m} C_rX_r + \sum_{r=0}^{q} d_rY_r$$

The program used to examine these models was basically that given on p. 404 of reference (29). It consists of a main routine named REGRE, a special input subroutine named DATA, and the following four subroutines from the Scientific Subroutine Package.

1. CORRE:- to find means, standard deviations and the correlation matrix.

2. ORDER:- to choose a dependent variable and a subset of independent variables from a larger set of variables

3. MTNV:- to invert the correlation matrix of the subset selected by ORDER.

4. MULTR:- to compute the regression coefficients and various confidence measures.

Very few modifications were necessary to the procedure given in the above reference to make these programs applicable to our models. Double precision was used throughout, as difficulties were experienced using single precision especially with the matrix inverting subroutine. The double precision versions of the above four subroutines are stored in the St. Andrews University Computing Laboratory under the names CLDCOR, CLDORD, CLDMIN and CLDMUL. At each value of $\lambda$ an 'observation' was read in. This consisted of the set

$$\left.\begin{array}{r} L \\ 2 \\ 1 \end{array}\right) \lambda, \lambda^2, \ldots \mu, \mu\lambda, \ldots\ldots$$

or

$$\left.\begin{array}{r} L \\ 2 \\ 1 \end{array}\right) X_1, X_2, \ldots\ldots, Y_o, Y_1, \ldots$$

(The variable to be estimated, and the variables on which it was supposed to depend linearly.) These observations were read into the computer one at a time by the input subroutine DATA. In practice the highest power of $\lambda$ taken was six, i.e. $X_6$ and $Y_6$, as it was not expected that any more complicated models would be required. The observations were read in according to format (4F20.15). In the set of original variables any variable can be designated as a dependent variable, and any number of variables can be specified as independent variables. The selection of a dependent variable and a set of independent variables can be performed over and over again using the same set of original variables. It can therefore be seen that this procedure gives great flexibility and enables us to determine several models quickly for fitting the data.

III.3 The Choice of the Order of Polynomial for $F_o(\lambda)$ and $F_1(\lambda)$

A guide to the values of n and m, (the orders of $F_o(\lambda)$ and $F_1(\lambda)$) was made by applying program REGRE to data from the Howarth solution for $u_1 = u_o(1-\xi)$, $\xi = x/c$, together with three values from the solution for $u_1 = u_o(1+\xi)$, (values given in Table 3.4). For these cases $\mu = 0$, so that the fitting of L and $l^2$ determines $F_o(\lambda)$ and $F_1(\lambda)$ alone. It must be remembered, however, that the forms yielded from the data from these solutions serve only as a guide to what happens over the larger range, but quite a good guide, as data from these solutions will be included in that covering the whole range. The orders of the polynomial estimates were increased until, for any value, the maximum modulus residual between L and its estimate was less than 0.002 and between $l^2$ and its estimate less than 0.0003. From Table 3.4, it can be seen that this happened when both $F_o(\lambda)$ and $F_1(\lambda)$ were of order four. From these results it was decided that even over the whole range of solutions there was very little to be gained in taking the polynomial models for $F_o(\lambda)$ and $F_1(\lambda)$ to be any more complicated than quintic, the extra degree being considered adequate to absorb the variations from solution to solution. Accordingly, in testing the models for L and $l^2$ over the whole range of data, the polynomials used to represent $F_0(\lambda)$ and $F_1(\lambda)$ were taken to be of order five.

## III.4 Application of program REGRE to data over the whole range

With $F_0(\lambda)$, $F_1(\lambda)$ taken to be polynomials of degree five, several models were tried for $G_0(\lambda)$ and $G_1(\lambda)$. In each case the sum of squared residuals was calculated and on the basis of the results obtained it was decided to use the forms with $G_0(\lambda)$ and $G_1(\lambda)$ both quintic. These gave

$$F_0(\lambda) = 0.44213 - 5.6718\lambda + 3.7148\lambda^2 + 4.3550\lambda^3 + 479.82\lambda^4 - 2740.1\lambda^5$$

$$G_0(\lambda) = 0.6571 + 2.335\lambda + 47.002\lambda^2 + 576.62\lambda^3 - 3145\lambda^4 - 20393\lambda^5$$

with Total Sum of Squared Residuals over 62 points = 0.000398

Root Mean Square Residual = 0.00253

Mean Modulus Residual = 0.00172

$$F_1(\lambda) = 0.04863 + 0.77696\lambda + 1.8359\lambda^2 - 2.3343\lambda^3 + 45.156\lambda^4 + 93.574\lambda$$

$$G_1(\lambda) = 0.1558 + 0.9714\lambda + 6.255\lambda^2 + 104.1\lambda^3 + 69.80\lambda^4 - 2741\lambda^5$$

with Total Sum of Squared Residuals over 62 points = 0.0000072

Root Mean Square Residual = 0.000342

Mean Modulus Residual = 0.000239

For the twenty-five points in Table 3.3 the corresponding quantities were:

For L: (Total Sum of Squared Residuals over 25 points = 0.000067)

Root Mean Square Residual = 0.00164

Mean Modulus Residual = 0.00105

For $1^2$: (Total Sum of Squared Residuals over 25 points = 0.0000025)

Root Mean Square Residual = 0.00031

Mean Modulus Residual = 0.00022

## III.5 Analysis for small values of $\lambda$

For small values of $\lambda$ in the range $-0.02 \leq \lambda \leq 0.02$ a separate analysis was carried out. In this case the leading coefficients in the functions $F_o(\lambda)$, $G_o(\lambda)$, $F_1(\lambda)$ and $G_1(\lambda)$ were related to ensure H, the shape parameter, was continuous at $\lambda = 0$. These conditions on the coefficients were derived as follows.

If we take

$$L = a_o + a_1\lambda + a_2\lambda^2 + \mu(b_o + b_1\lambda) + \ldots\ldots$$

$$1^2 = c_o + c_1\lambda + c_2\lambda^2 + \mu(d_o + d_1\lambda) + \ldots\ldots$$

$$1 = c_o^{\frac{1}{2}}\left[1 + \frac{c_1}{c_o}\lambda + \frac{c_2}{c_o}\lambda^2 + \mu\left(\frac{d_o}{c_o} + \frac{d_1}{c_o}\lambda\right)\right]^{\frac{1}{2}} + \ldots\ldots$$

$$= c_o^{\frac{1}{2}}\left[1 + \tfrac{1}{2}\left[\frac{c_1}{c_o}\lambda + \frac{c_2}{c_o}\lambda^2 + \mu\left(\frac{d_o}{c_o} + \frac{d_1}{c_o}\lambda\right)\right] + \frac{\frac{1}{2}\cdot-\frac{1}{2}}{2!}\left[\frac{c_1}{c_o}\lambda + \frac{c_2}{c_o}\lambda^2 + \mu\left(\frac{d_o}{c_o} + \frac{d_1}{c_o}\lambda\right)\right]\right]$$

$$= c_o^{\frac{1}{2}} + \tfrac{1}{2}\frac{c_1}{c_o^{\frac{1}{2}}}\lambda + \tfrac{1}{2}\frac{c_2}{c_o^{\frac{1}{2}}}\lambda^2 + \mu\left(\frac{d_o}{2c_o^{\frac{1}{2}}}\right) + \tfrac{1}{2}\mu\lambda\frac{d_1}{c_o^{\frac{1}{2}}} - \frac{1}{8}\frac{c_1^2}{c_o^{3/2}}\lambda^2 - \frac{1}{8}\frac{c_2^2}{c_o^{3/2}}\lambda^4$$

$$- \frac{1}{8}\cdot 2\cdot \frac{c_1}{c_o^{\frac{1}{2}}}\lambda\mu\left(\frac{d_o}{c_o} + \frac{d_1}{c_o}\lambda\right) + \ldots\ldots\ldots$$

$$= c_o^{\frac{1}{2}} + \tfrac{1}{2}\frac{c_1}{c_o^{\frac{1}{2}}}\lambda + \left(\tfrac{1}{2}\frac{c_2}{c_o^{\frac{1}{2}}} - \frac{1}{8}\frac{c_1^2}{c_o^{3/2}}\right)\lambda^2 + \mu\left(\frac{d_o}{2c_o^{\frac{1}{2}}}\right) + \mu\lambda\left(\frac{d_1}{2c_o^{\frac{1}{2}}} - \frac{1}{4}\frac{c_1 d_o}{c_o^{3/2}}\right) + \ldots$$

$$\therefore\ 21 = 2c_o^{\frac{1}{2}} + \frac{c_1}{c_o^{\frac{1}{2}}}\lambda + \left(\frac{c_2}{c_o^{\frac{1}{2}}} - \frac{1}{4}\frac{c_1^2}{c_o^{3/2}}\right)\lambda^2 + \mu\left(\frac{d_o}{c_o^{\frac{1}{2}}}\right) + \mu\lambda\left(\frac{d_1}{c_o^{\frac{1}{2}}} - \frac{c_1 d_o}{2c_o^{3/2}}\right) + \ldots\ldots$$

$$H = \frac{21-L-4\lambda}{2\lambda}$$

$$= \frac{2c_o^{\frac{1}{2}} + \frac{c_1}{c_o^{\frac{1}{2}}} + \left(\frac{c_2}{c_o^{\frac{1}{2}}} - \frac{1}{4}\frac{c_1^2}{c_o^{3/2}}\right)\lambda^2 + \mu\left(\frac{d_o}{c_o^{\frac{1}{2}}}\right) + \mu\lambda\left(\frac{d_1}{c_o^{\frac{1}{2}}} - \frac{c_1 d_o}{2c_o^{3/2}}\right) - }{2\lambda} \quad \frac{(a_o + a_1\lambda + a_2\lambda^2 + \mu(b_o + b_1\lambda)) - 4\lambda}{2\lambda}$$

$$= \frac{(2c_o^{\frac{1}{2}}-a_o)+(\frac{c_1}{c_o^{\frac{1}{2}}} - a_1-4)\lambda+(\frac{c_2}{c_o^{\frac{1}{2}}} - \frac{1}{4}\frac{c_1^2}{c_o^{3/2}} - a_2)\lambda^2+\mu(\frac{d_o}{c_o^{\frac{1}{2}}} - b_o)+\mu\lambda(\frac{d_1}{c_o^{\frac{1}{2}}} - \frac{c_1 d_o}{2c_o^{3/2}} - b_1)}{2} + \cdots\cdots$$

$$= \frac{(2c_o^{\frac{1}{2}} - a_o)}{2\lambda} + \tfrac{1}{2}(\frac{c_1}{c_o^{\frac{1}{2}}} - a_1 - 4)+\tfrac{1}{2}(\frac{c_2}{c_o^{\frac{1}{2}}} - \frac{1}{4}\frac{c_1^2}{c_o^{3/2}} - a_2)\lambda+ \frac{\mu}{2\lambda}(\frac{d_o}{c_o^{\frac{1}{2}}} - b_o) + \mu\cdot\tfrac{1}{2}(\frac{d_1}{c_o^{\frac{1}{2}}} - \frac{c_1 d_o}{2c_o^{3/2}} - b_1)$$

$$= H_o + H_1\lambda + \epsilon_o\mu + \cdots\cdots$$

To prevent a discontinuity in H when $\lambda = 0$ we must have

$$2c_o^{\frac{1}{2}} - a_o = 0 \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (1)$$

$$\frac{d_o}{c_o^{\frac{1}{2}}} - b_o = 0 \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (2)$$

Also given $a_o$, $a_1$, $a_2$, $b_o$, $b_1$ and $c_o$, $c_1$, $c_2$, $d_o$ and $d_1$ the leading terms in an expansion for H for small $\lambda$ are determined by

$$H_o = \tfrac{1}{2}(\frac{c_1}{c_o^{\frac{1}{2}}} - a_1 - 4) \quad \cdots\cdots\cdots\cdots \quad (3)$$

$$H_1 = \tfrac{1}{2}(\frac{c_2}{c_o^{\frac{1}{2}}} - \frac{1}{4}\frac{c_1^2}{c_o^{3/2}} - a_2) \quad \cdots\cdots\cdots \quad (4)$$

$$\epsilon_o = \tfrac{1}{2}(\frac{d_1}{c_o^{\frac{1}{2}}} - \frac{c_1 d_o}{2c_o^{3/2}} - b_1) \quad \cdots\cdots\cdots \quad (5)$$

The numerical values for these coefficients were obtained by applying the program REGRE to the sixteen points in Table 3.2. From an examination of the sum of squared residuals for several models it was decided that in this region $F_o(\lambda)$ and $F_1(\lambda)$ should be represented by quartics in $\lambda$ and $G_o(\lambda)$ and $G_1(\lambda)$ by cubics. The values of H were also examined and to them was fitted a model, quadratic in $\lambda$ plus $\mu$ times a linear function of $\lambda$. The leading coefficients were then slightly altered so that relationships (1) to (5) were satisfied. The forms which resulted were

$$F_{\circ}(\lambda) = 0.441117 - 5.6851\lambda + 6.\dot{8}37\lambda^2 + 258.65\lambda^3 - 5721\lambda^4$$

$$G_{\circ}(\lambda) = 0.704761 + 3.70\lambda + 29.639\lambda^2 - 3135\lambda^3$$

with Total Sum of Squared Residuals = 0.000048

Root Mean Square Residual = 0.0017

Mean Modulus Residual = 0.0014

$$F_1(\lambda) = 0.048646 + 0.7687\lambda + 2.100\lambda^2 + 56.90\lambda^3 - 1234\lambda^4$$

$$G_1(\lambda) = 0.155441 + 1.3275\lambda + 8.008\lambda^2 - 671\lambda^3$$

with Total Sum of Squared Residuals = 0.0000007

Root Mean Square Residual = 0.00022

Mean Modulus Residual = 0.00017

and

$$H(\lambda,\mu) = 2.585 - 5.542\lambda + 26.78\lambda^2 + \mu(1.6246 - 5.11\lambda)$$

with Total Sum of Square Residuals = 0.0004

Root Mean Square Residual = 0.0050

Mean Modulus Residual = 0.0045

## III.6 Construction of the Tables

From the forms which have been given in the previous sections the numerical tables for the functions $F_{\circ}(\lambda)$, $G_{\circ}(\lambda)$, $F_1(\lambda)$ and $G_1(\lambda)$ were calculated as follows. For the range $|\lambda| \geqslant 0.025$ the values were obtained from the forms derived using the sixty-two points given in Table 3.1, which covered the whole range of $\lambda$. For the range $|\lambda| \leqslant 0.015$ the values were obtained from the forms yielded using the data of Tables 3.1 and 3.2 were found to be in good agreement. To decide on the values to be chosen for inclusion in the tables the following procedure was adopted. In each case the relevant function $F_o(\lambda)$, $G_o(\lambda)$, $F_1(\lambda)$ and $G_1(\lambda)$ (using both the form yielded by the data in Table 3.1 and the form yielded using Table 3.2) was plotted on a greatly magnified scale over the range $0.015 \leq |\lambda| \leq 0.025$ and a smooth curve drawn through the region of overlap. The required values were then read off from the graph. In practice the only $\lambda$ values in the tables which are now given, which required this treatment were those corresponding to $\lambda = \pm 0.02$. The resulting values are given in Table 3.5.

Table 3.1: Data for use with program REGRE

| Solution | $\lambda$ | $\mu$ | L | $l^2$ | H |
|---|---|---|---|---|---|
| $u_0\sin\xi$ | -0.116843 | -0.205621 | 1.29046 | 0.0 | 3.521 |
| $u_0\sin\xi$ | -0.099811 | -0.183026 | 1.15427 | 0.005097 | 3.067 |
| $u_0(1-\xi^2)$ | -0.0916 | -0.0531 | 1.055232 | 0.0 | 3.760 |
| $u_0(1-\xi^2)$ | -0.0864 | -0.0496 | 1.01347 | 0.001706 | 3.387 |
| $u_0(1-\xi)$ | -0.0841 | 0.0 | 0.980202 | 0.0 | 3.8276 |
| $u_0(1-\xi)$ | -0.07618 | 0.0 | 0.920220 | 0.00258 | 3.3732 |
| P.G. Williams | -0.07585 | 0.04293 | 0.89908 | 0.000008 | 3.8898 |
| $u_0\sin\xi$ | -0.074505 | -0.151379 | 0.971718 | 0.015744 | 2.837 |
| $u_0(1-\xi^2)$ | -0.0742 | -0.0412 | 0.922615 | 0.006304 | 3.147 |
| P.G. Williams | -0.07010 | 0.03708 | 0.85852 | 0.00233 | 3.4349 |
| $u_0\xi^n$ | -0.068148 | 0.056002 | 0.821763 | 0.0 | 4.02923 |
| $u_0(1-\xi)$ | -0.06452 | 0.0 | 0.836718 | 0.00768 | 3.1260 |
| $u_0\xi^n$ | -0.063202 | 0.050030 | 0.791874 | 0.002460 | 3.47982 |
| $u_0\sin\xi$ | -0.062797 | -0.137259 | 0.891044 | 0.021457 | 2.762 |
| $u_0\xi^n$ | -0.057994 | 0.044093 | 0.760230 | 0.005320 | 3.29668 |
| $u_0(1-\xi^2)$ | -0.0557 | -0.0295 | 0.794027 | 0.014835 | 2.941 |
| $u_0(1-\xi)$ | -0.05336 | 0.0 | 0.76505 | 0.01350 | 2.9913 |
| P.G. Williams | -0.05250 | 0.02148 | 0.74746 | 0.01187 | 3.0429 |
| $u_0\xi^n$ | -0.048811 | 0.034539 | 0.707500 | 0.011071 | 3.09170 |
| $u_0(1-\xi)$ | -0.04368 | 0.0 | 0.703911 | 0.01920 | 2.8852 |
| $u_0\xi^n$ | -0.040643 | 0.026904 | 0.661545 | 0.016652 | 2.96348 |
| $u_0(1-\xi^2)$ | -0.0392 | -0.0197 | 0.684618 | 0.023593 | 2.814 |
| $u_0\sin\xi$ | -0.035808 | -0.106668 | 0.712631 | 0.036408 | 2.622 |
| $u_0(1-\xi)$ | -0.03460 | 0.0 | 0.646576 | 0.02459 | 2.8118 |
| P.G. Williams | -0.03290 | 0.00870 | 0.63069 | 0.02458 | 2.8200 |
| $u_0\xi^n$ | -0.026527 | 0.015481 | 0.583594 | 0.027040 | 2.80112 |
| $u_0(1-\xi^2)$ | -0.0247 | -0.0119 | 0.592417 | 0.032256 | 2.721 |
| $u_0(1-\xi)$ | -0.01877 | 0.0 | 0.549378 | 0.034715 | 2.7080 |
| $u_0\sin\xi$ | -0.017497 | -0.087143 | 0.596018 | 0.047743 | 2.544 |
| $u_0(1-\xi^2)$ | -0.0116 | -0.0054 | 0.511057 | 0.040642 | 2.649 |
| $u_0\sin\xi$ | -0.008143 | -0.077713 | 0.537908 | 0.053941 | 2.507 |
| $u_0(1-\xi)$ | -0.00578 | 0.0 | 0.474889 | 0.04441 | 2.6184 |
| $u_0(1+\xi)$ | 0.0 | 0.0 | 0.441056 | 0.048632 | 2.59105 |
| $u_0(1+\xi)$ | 0.010 | 0.0 | 0.384513 | 0.056475 | 2.53887 |
| $u_0\sin\xi$ | 0.016475 | -0.053972 | 0.388457 | 0.071323 | 2.421 |
| $u_0\xi^n$ | 0.018962 | -0.006472 | 0.341327 | 0.065347 | 2.4809 |
| $u_0(1+\xi)$ | 0.020 | 0.0 | 0.329144 | 0.064707 | 2.49022 |
| $u_0\sin\xi$ | 0.025882 | -0.045523 | 0.332969 | 0.078459 | 2.390 |
| $u_0(1+\xi)$ | 0.030 | 0.0 | 0.274946 | 0.073338 | 2.4444 |
| $u_0\xi^n$ | 0.03333 | -0.008887 | 0.266646 | 0.078588 | 2.4108 |
| $u_0\sin\xi$ | 0.03532 | -0.037336 | 0.277916 | 0.085842 | 2.361 |
| $u_0(1+\xi)$ | 0.040 | 0.0 | 0.221952 | 0.082377 | 2.4009 |
| $u_0\xi^n$ | 0.044639 | -0.009299 | 0.208304 | 0.089314 | 2.3617 |

| Solution | $\lambda$ | $\mu$ | L | $l^2$ | H |
|---|---|---|---|---|---|
| $u_0 \sin\xi$ | 0.046493 | -0.028047 | 0.213769 | 0.094900 | 2.327 |
| $u_0(1+\xi)$ | 0.050 | 0.0 | 0.17024 | 0.091856 | 2.359 |
| $u_0\xi^n$ | 0.053785 | -0.008678 | 0.161353 | 0.098162 | 2.3252 |
| $u_0 \sin\xi$ | 0.055880 | -0.020659 | 0.160784 | 0.102834 | 2.300 |
| $u_0\xi^n$ | 0.061345 | -0.007526 | 0.122689 | 0.105585 | 2.29694 |
| $u_0 \sin\xi$ | 0.068476 | -0.011373 | 0.091309 | 0.113951 | 2.263 |
| $u_0\xi^n$ | 0.076429 | -0.003505 | 0.045827 | 0.120661 | 2.24492 |
| $u_0 \sin\xi$ | 0.082227 | -0.002018 | 0.016914 | 0.126699 | 2.226 |
| $u_0\xi^n$ | 0.085465 | 0.0 | 0.0 | 0.129844 | 2.21623 |
| $u_0(\xi+\xi^3)$ | 0.087 | 0.00094 | -0.008050 | 0.131342 | 2.21192 |
| $u_0\xi^n$ | 0.088977 | 0.001583 | -0.017795 | 0.133441 | 2.20552 |
| $u_0(\xi+\xi^3)$ | 0.090 | 0.002700 | -0.023658 | 0.134310 | 2.20347 |
| $u_0\xi^n$ | 0.092793 | 0.003444 | -0.037117 | 0.137366 | 2.19416 |
| $u_0(\xi+\xi^3)$ | 0.094 | 0.004880 | -0.044157 | 0.138358 | 2.19196 |
| $u_0\xi^n$ | 0.096955 | 0.005640 | -0.058173 | 0.141666 | 2.18207 |
| $u_0(\xi+\xi^3)$ | 0.098427 | 0.006459 | -0.065616 | 0.143191 | 2.17786 |
| $u_0\xi^n$ | 0.101512 | 0.008244 | -0.081209 | 0.146394 | 2.16918 |
| $u_0\xi^n$ | 0.118204 | 0.019561 | -0.165485 | 0.163888 | 2.12485 |
| $u_0\xi^n$ | 0.141497 | 0.040043 | -0.282995 | 0.188663 | 2.06969 |

Table 3.2: Data for small values of lambda

| Solution | $\lambda$ | $\mu$ | L | $l^2$ | H |
|---|---|---|---|---|---|
| $u_0(1+\xi)$ | 0.020 | 0.0 | 0.329144 | 0.064707 | 2.49022 |
| $u_0\xi^n$ | 0.0189619 | -0.006472 | 0.341327 | 0.065347 | 2.4809 |
| $u_0 \sin\xi$ | 0.016475 | -0.053972 | 0.388457 | 0.071323 | 2.421 |
| $u_0(1+\xi)$ | 0.015 | 0.0 | 0.356681 | 0.060542 | 2.51415 |
| $u_0(1+\xi)$ | 0.010 | 0.0 | 0.384513 | 0.056475 | 2.53887 |
| $u_0(1+\xi)$ | 0.005 | 0.0 | 0.412635 | 0.052505 | 2.56447 |
| $u_0 \sin\xi$ | 0.002793 | -0.066904 | 0.470857 | 0.061457 | 2.468 |
| $u_0(1+\xi)$ | 0.0 | 0.0 | 0.441056 | 0.048632 | 2.59105 |
| $u_0(1-\xi)$ | -0.00578 | 0.0 | 0.474889 | 0.044415 | 2.6184 |
| $u_0 \sin\xi$ | -0.008143 | -0.077713 | 0.537908 | 0.053941 | 2.507 |
| $u_0(1-\xi^2)$ | -0.0116 | -0.0054 | 0.511057 | 0.040642 | 2.649 |
| $u_0(1-\xi)$ | -0.01210 | 0.0 | 0.512379 | 0.03995 | 2.6545 |
| P.G. Williams | -0.013160 | 0.001430 | 0.516230 | 0.038675 | 2.66970 |
| $u_0 \sin\xi$ | -0.014251 | -0.083887 | 0.575779 | 0.049870 | 2.531 |
| $u_0 \sin\xi$ | -0.017497 | -0.087143 | 0.596018 | 0.047743 | 2.544 |
| $u_0(1-\xi)$ | -0.01877 | 0.0 | 0.549378 | 0.034715 | 2.7080 |

Table 3.3: Data used for checking forms

| Solution | $\lambda$ | $\mu$ | L | $l^2$ | H |
|---|---|---|---|---|---|
| $u_1 = u_0(\xi+\xi^3)$ | 0.086 | 0.000330 | -0.002811 | 0.130365 | 2.21473 |
| | 0.088 | 0.001539 | -0.013271 | 0.132326 | 2.20911 |
| | 0.091 | 0.003275 | -0.028818 | 0.135311 | 2.20062 |
| | 0.093 | 0.004359 | -0.039071 | 0.137335 | 2.19488 |
| $u_0(1+\xi)$ | 0.005 | 0.0 | 0.412635 | 0.052505 | 2.56447 |
| | 0.015 | 0.0 | 0.356681 | 0.060542 | 2.51415 |
| | 0.025 | 0.0 | 0.301890 | 0.068969 | 2.4670 |
| | 0.035 | 0.0 | 0.248285 | 0.077802 | 2.4225 |
| $u_0 \sin\xi$ | -0.091366 | -0.172255 | 1.091726 | 0.008338 | 2.975 |
| | -0.106238 | -0.191430 | 1.203528 | 0.002893 | 3.158 |
| | -0.111425 | -0.198271 | 1.244467 | 0.001314 | 3.259 |
| | -0.115267 | -0.203435 | 1.276556 | 0.000325 | 3.381 |
| | -0.116431 | -0.205077 | 1.286746 | 0.000078 | 3.450 |
| $u_0\xi^n$ | 0.067701 | -0.006111 | 0.090274 | 0.111898 | 2.2743 |
| | 0.077800 | -0.003026 | 0.038905 | 0.122049 | 2.2404 |
| | 0.082222 | -0.001352 | 0.016444 | 0.126537 | 2.2263 |
| | 0.125067 | 0.025027 | -0.200108 | 0.171149 | 2.107841 |
| $u_0(1-\xi)$ | -0.01210 | 0.0 | 0.512379 | 0.03995 | 2.6545 |
| | -0.02624 | 0.0 | 0.594503 | 0.02971 | 2.7592 |
| $u_0(1-\xi^2)$ | 0.0284 | 0.0111 | 0.276331 | 0.070225 | 2.466 |
| | 0.0492 | 0.0173 | 0.160319 | 0.087616 | 2.387 |
| P.G. Williams' Solution | -0.01316 | 0.00143 | 0.51623 | 0.03867 | 2.6697 |
| | -0.04696 | 0.01734 | 0.71392 | 0.01529 | 2.9682 |
| | -0.06872 | 0.03572 | 0.84939 | 0.00298 | 3.3858 |
| | -0.07513 | 0.04217 | 0.89347 | 0.00024 | 3.7390 |

Table 3.4: Results of applying REGRE to Howarth data

Howarth Data

$u_1 = u_\circ(1-\xi)$, $\mu = 0.0$

| $\lambda$ | L | $l^2$ | H |
|---|---|---|---|
| 0.00000 | 0.441056 | 0.04863 | 2.591053 |
| -0.00578 | 0.474889 | 0.04441 | 2.6184 |
| -0.01210 | 0.512379 | 0.03995 | 2.6545 |
| -0.01877 | 0.549378 | 0.03471 | 2.7080 |
| -0.02624 | 0.594503 | 0.02971 | 2.7592 |
| -0.03460 | 0.646576 | 0.02459 | 2.8118 |
| -0.04368 | 0.703911 | 0.01920 | 2.8852 |
| -0.05336 | 0.765051 | 0.01350 | 2.9913 |
| -0.06452 | 0.836718 | 0.00768 | 3.1260 |
| -0.07618 | 0.920220 | 0.00258 | 3.3732 |
| -0.08410 | 0.980202 | 0.00000 | 3.8276 |

$u_1 = u_\circ(1+\xi)$, $\mu = 0$

| $\lambda$ | L | $l^2$ | H |
|---|---|---|---|
| 0.005 | 0.412635 | 0.052505 | 2.56447 |
| 0.015 | 0.356681 | 0.060542 | 2.51415 |
| 0.085465 | 0.000000 | 0.129845 | 2.2165 |

Results, Order(s) of $F_\circ(\lambda)$, $F_1(\lambda)$ v Max Mod Residual

| Order of $F_\circ(\lambda)$ | Max. Mod. Res | Order of $F_1(\lambda)$ | Max. Mod. Res |
|---|---|---|---|
| 1 | 0.044 | 1 | 0.013 |
| 2 | 0.005 | 2 | 0.000598 |
| 3 | 0.0034 | 3 | 0.000375 |
| 4 | 0.0015 | 4 | 0.000295 |
| 5 | 0.0014 | 5 | 0.000292 |

Table 3.5: The values of $F_o(\lambda)$, $G_o(\lambda)$, $F_1(\lambda)$, $G_1(\lambda)$

| $\lambda$ | $F_o(\lambda)$ | $G_o(\lambda)$ | $F_1(\lambda)$ | $G_1(\lambda)$ |
|---|---|---|---|---|
| -0.12 | 1.33638 | -0.08736 | -0.00710 | 0.03210 |
| -0.11 | 1.21954 | 0.06949 | -0.00641 | 0.04044 |
| -0.10 | 1.11747 | 0.20645 | -0.00479 | 0.05150 |
| -0.09 | 1.02715 | 0.32141 | -0.00231 | 0.06392 |
| -0.08 | 0.94603 | 0.41391 | 0.00096 | 0.07666 |
| -0.07 | 0.87197 | 0.48496 | 0.00497 | 0.08903 |
| -0.06 | 0.80319 | 0.53677 | 0.00964 | 0.10059 |
| -0.05 | 0.73829 | 0.57250 | 0.01492 | 0.11115 |
| -0.04 | 0.67618 | 0.59604 | 0.02074 | 0.12075 |
| -0.03 | 0.61596 | 0.61173 | 0.02707 | 0.12960 |
| -0.02 | 0.55575 | 0.63625 | 0.03364 | 0.13732 |
| -0.01 | 0.49834 | 0.67386 | 0.04110 | 0.14364 |
| 0.00 | 0.44112 | 0.70476 | 0.04865 | 0.15544 |
| 0.01 | 0.38515 | 0.74159 | 0.05659 | 0.16885 |
| 0.02 | 0.33051 | 0.74875 | 0.06501 | 0.17965 |
| 0.03 | 0.27576 | 0.78198 | 0.07357 | 0.19337 |
| 0.04 | 0.22243 | 0.85247 | 0.08262 | 0.21122 |
| 0.05 | 0.17051 | 0.93740 | 0.09209 | 0.23260 |
| 0.06 | 0.12022 | 1.03434 | 0.10201 | 0.25786 |
| 0.07 | 0.07172 | 1.13885 | 0.11245 | 0.28722 |
| 0.08 | 0.02507 | 1.24430 | 0.12350 | 0.32072 |
| 0.09 | -0.01976 | 1.34156 | 0.13524 | 0.35817 |
| 0.10 | -0.06296 | 1.41881 | 0.14780 | 0.39916 |
| 0.11 | -0.10490 | 1.46126 | 0.16132 | 0.44297 |
| 0.12 | -0.14615 | 1.45094 | 0.17596 | 0.48859 |
| 0.13 | -0.18755 | 1.36641 | 0.19190 | 0.53466 |
| 0.14 | -0.23020 | 1.18254 | 0.20936 | 0.57944 |

Table 3.6: Comparison of Values of L

$L_C$ = Value using Original Tables
$L_L$ = Value using New Tables
$L_E$ = Exact Value of L

| $\frac{u_1}{u_o}$ | $\lambda$ | $L_E$ | $L_C$ | $L_L$ | $\|L_E-L_C\|$ | $\|L_E-L_L\|$ |
|---|---|---|---|---|---|---|
| | 0.0855 | 0.0 | 0.0 | 0.000 | 0.000 | 0.0 |
| | 0.0613 | 0.123 | 0.121 | 0.121 | 0.002 | 0.002 |
| | 0.0333 | 0.267 | 0.266 | 0.265 | 0.001 | 0.002 |
| $\xi^n$ | 0.0 | 0.441 | 0.441 | 0.441 | 0.000 | 0.0 |
| | -0.0265 | 0.584 | 0.586 | 0.585 | 0.002 | 0.001 |
| | -0.0488 | 0.708 | 0.715 | 0.711 | 0.007 | 0.003 |
| | -0.0681 | 0.822 | 0.835 | 0.831 | 0.013 | 0.009 |
| | -0.0121 | 0.512 | 0.511 | 0.510 | 0.002 | 0.002 |
| | -0.0346 | 0.646 | 0.645 | 0.643 | 0.001 | 0.003 |
| $1-\xi$ | -0.0645 | 0.837 | 0.836 | 0.834 | 0.001 | 0.003 |
| | -0.0841 | 0.980 | 0.973 | 0.978 | 0.007 | 0.002 |
| | 0.0685 | 0.091 | 0.091 | 0.092 | 0.000 | 0.001 |
| | 0.0165 | 0.388 | 0.388 | 0.388 | 0.000 | 0.000 |
| $\sin\xi$ | -0.0244 | 0.639 | 0.641 | 0.641 | 0.002 | 0.002 |
| | -0.0584 | 0.861 | 0.859 | 0.864 | 0.002 | 0.003 |
| | -0.0865 | 1.056 | 1.057 | 1.057 | 0.001 | 0.001 |
| | -0.1168 | 1.290 | 1.292 | 1.290 | 0.002 | 0.000 |
| | -0.0392 | 0.685 | 0.683 | 0.683 | 0.002 | 0.002 |
| $1-\xi^2$ | -0.0742 | 0.923 | 0.920 | 0.921 | 0.003 | 0.002 |
| | -0.0916 | 1.055 | 1.050 | 1.057 | 0.005 | 0.002 |
| | 0.01 | 0.384 | 0.385 | 0.385 | 0.001 | 0.001 |
| $1+\xi$ | 0.02 | 0.329 | 0.330 | 0.330 | 0.001 | 0.001 |
| | 0.03 | 0.275 | 0.276 | 0.276 | 0.001 | 0.001 |
| | 0.04 | 0.222 | 0.224 | 0.222 | 0.002 | 0.000 |

Table 3.7: Comparison of Values of $l^2$

$l_C^2$ = Value using Original Tables

$l_L^2$ = Value using New Tables

$l_E^2$ = Exact Value of $l^2$

| $\frac{u_1}{u_o}$ | $\lambda$ | $l_E^2$ | $l_C^2$ | $l_L^2$ | $\lvert l_E^2 - l_C^2 \rvert$ | $\lvert l_E^2 - l_L^2 \rvert$ |
|---|---|---|---|---|---|---|
| | 0.0855 | 0.1298 | 0.1296 | 0.1298 | 0.0002 | 0.0000 |
| | 0.0613 | 0.1056 | 0.1055 | 0.1054 | 0.0001 | 0.0002 |
| | 0.0333 | 0.0786 | 0.0786 | 0.0783 | 0.0000 | 0.0003 |
| $\xi^n$ | 0.0 | 0.0486 | 0.0487 | 0.0486 | 0.0001 | 0.0000 |
| | -0.0265 | 0.0270 | 0.0274 | 0.0273 | 0.0004 | 0.0003 |
| | -0.0488 | 0.0119 | 0.0119 | 0.0117 | 0.0000 | 0.0002 |
| | -0.0681 | 0.0 | 0.0007 | 0.0007 | 0.0007 | 0.0007 |
| | -0.0121 | 0.0399 | 0.0396 | 0.0395 | 0.0003 | 0.0004 |
| $1-\xi$ | -0.0346 | 0.0246 | 0.0241 | 0.0241 | 0.0005 | 0.0005 |
| | -0.0645 | 0.0077 | 0.0073 | 0.0075 | 0.0004 | 0.0002 |
| | -0.0841 | 0.0 | -0.0002 | -0.0005 | 0.0002 | 0.0005 |
| | 0.0685 | 0.1140 | 0.1139 | 0.1140 | 0.0001 | 0.0000 |
| | 0.0165 | 0.0713 | 0.0712 | 0.0713 | 0.0001 | 0.0000 |
| $\sin\xi$ | -0.0244 | 0.0434 | 0.0429 | 0.0435 | 0.0005 | 0.0001 |
| | -0.0584 | 0.0238 | 0.0235 | 0.0240 | 0.0003 | 0.0002 |
| | -0.0865 | 0.0103 | 0.0098 | 0.0101 | 0.0005 | 0.0002 |
| | -0.1168 | 0.0 | -0.0004 | 0.0001 | 0.0004 | 0.0001 |
| | -0.0392 | 0.0236 | 0.0234 | 0.0236 | 0.0002 | 0.0000 |
| $1-\xi^2$ | -0.0742 | 0.0063 | 0.0063 | 0.0067 | 0.0000 | 0.0004 |
| | -0.0916 | 0.0 | 0.0004 | 0.0005 | 0.0004 | 0.0005 |
| | 0.01 | 0.0565 | 0.0567 | 0.0566 | 0.0002 | 0.0001 |
| $1+\xi$ | 0.02 | 0.0647 | 0.0651 | 0.0650 | 0.0004 | 0.0003 |
| | 0.03 | 0.0733 | 0.0738 | 0.0736 | 0.0005 | 0.0003 |
| | 0.04 | 0.0824 | 0.0830 | 0.0826 | 0.0006 | 0.0002 |

Table 3.8: Comparison of Values of H

$H_C$ = Value using Original Tables

$H_L$ = Value using New Tables

$H_E$ = Exact Value of H

| $\frac{u_1}{u_0}$ | $\lambda$ | $H_E$ | $H_C$ | $H_L$ | $\lvert H_E-H_C\rvert$ | $\lvert H_E-H_L\rvert$ |
|---|---|---|---|---|---|---|
| $\xi^n$ | 0.0855 | 2.216 | 2.211 | 2.214 | 0.005 | 0.002 |
| | 0.0613 | 2.297 | 2.312 | 2.308 | 0.015 | 0.011 |
| | 0.0333 | 2.411 | 2.425 | 2.424 | 0.014 | 0.013 |
| | 0.0 | 2.591 | —— | 2.585 | —— | 0.006 |
| | -0.0265 | 2.801 | 2.810 | 2.803 | 0.009 | 0.002 |
| | -0.0488 | 3.092 | 3.090 | 3.068 | 0.002 | 0.024 |
| $1-\xi$ | -0.0121 | 2.654 | 2.670 | 2.656 | 0.016 | 0.002 |
| | -0.0346 | 2.812 | 2.834 | 2.805 | 0.022 | 0.007 |
| | -0.0645 | 3.126 | 3.156 | 3.127 | 0.030 | 0.001 |
| $\sin\xi$ | 0.0685 | 2.263 | 2.263 | 2.258 | 0.000 | 0.005 |
| | 0.0165 | 2.421 | 2.414 | 2.425 | 0.007 | 0.004 |
| | -0.0244 | 2.572 | 2.647 | 2.587 | 0.075 | 0.015 |
| | -0.0584 | 2.736 | 2.730 | 2.744 | 0.006 | 0.008 |
| | -0.0865 | 2.930 | 2.965 | 2.948 | 0.035 | 0.018 |
| $1-\xi^2$ | -0.0392 | 2.814 | 2.809 | 2.793 | 0.005 | 0.021 |
| | -0.0742 | 3.147 | 3.130 | 3.103 | 0.017 | 0.044 |
| $1+\xi$ | 0.01 | 2.539 | 2.562 | 2.532 | 0.023 | 0.007 |
| | 0.02 | 2.490 | 2.507 | 2.485 | 0.017 | 0.005 |
| | 0.03 | 2.444 | 2.455 | 2.443 | 0.011 | 0.001 |
| | 0.04 | 2.401 | 2.402 | 2.410 | 0.001 | 0.009 |

FIG 3.1 COMPARISON OF THE VALUES OF H AS GIV
THE METHOD, WITH THE EXACT VALUES
FALKNER SKAN SOLUTIONS.

EXACT SOLUTION SHOWN -----

| $\lambda$ | $H_{EXACT}$ | $H_{METHOD}$ |
|---|---|---|
| 0.0855 | 2.216 | 2.214 |
| 0.0613 | 2.297 | 2.308 |
| 0.0333 | 2.411 | 2.424 |
| 0.0 | 2.591 | 2.585 |
| −0.0265 | 2.801 | 2.803 |
| −0.0406 | 2.964 | 2.951 |
| −0.0488 | 3.092 | 3.068 |
| −0.0580 | 3.297 | 3.254 |
| −0.0632 | 3.480 | 3.420 |
| −0.0681 | 4.029 | 3.742 |

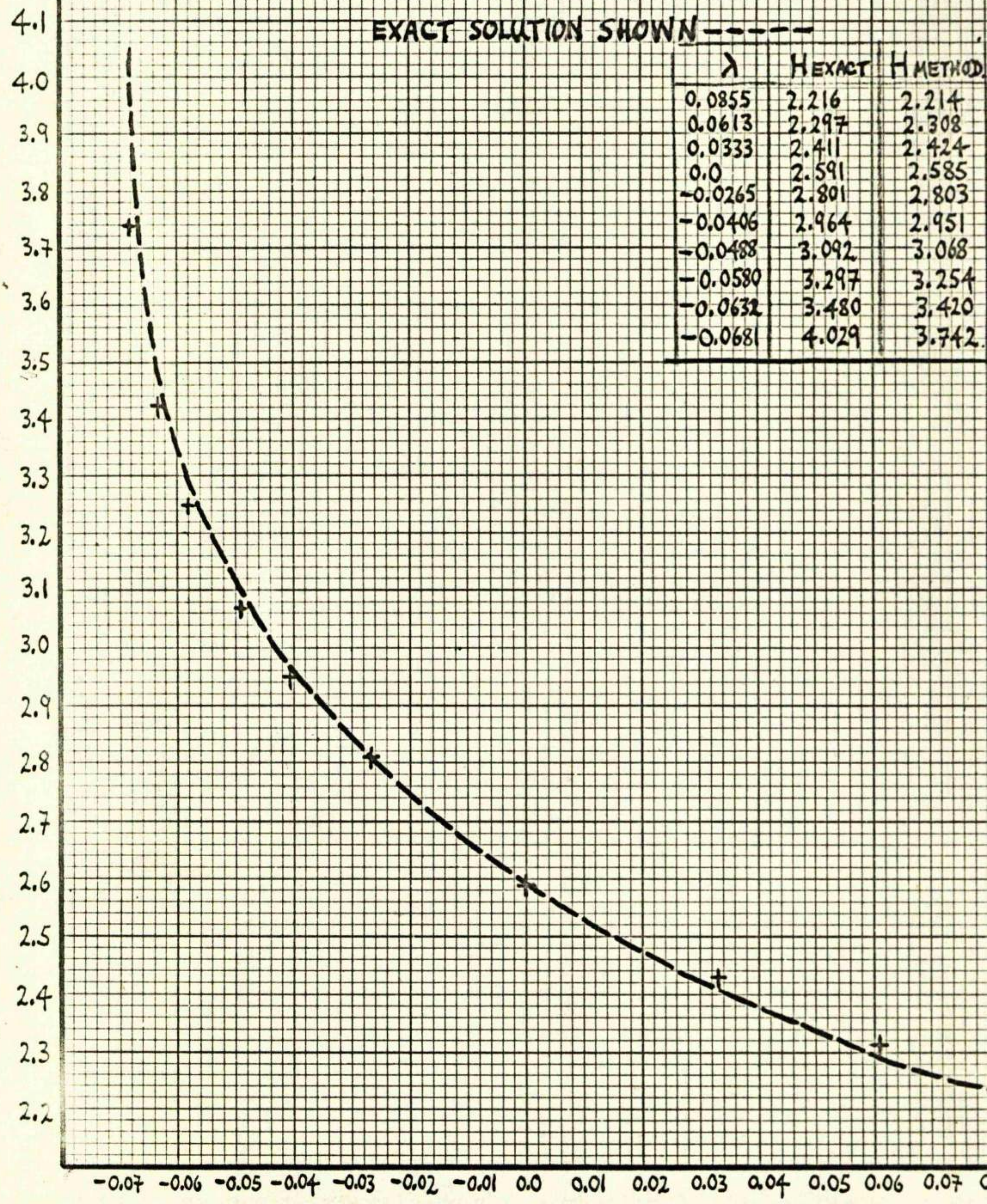
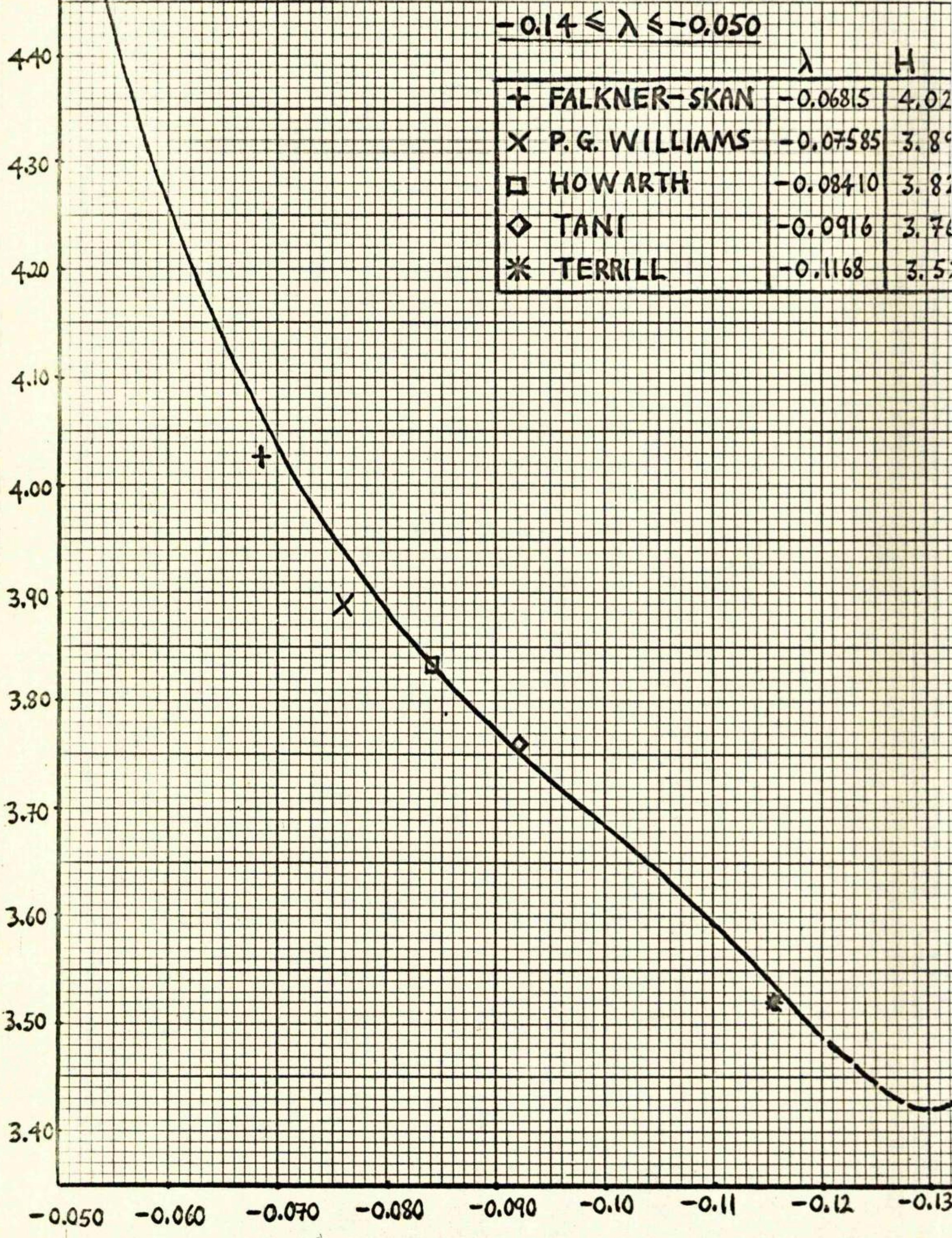


FIG 3.2 GRAPH OF H versus λ AT SEPARATION



Discussion of Tables 3.6, 3.7 and 3.8

The main improvements to the tables are that the range over which they are defined has been extended and that the values have been adjusted to give an analytic form for the shape parameter for small values of $\lambda$. Further comments on the tables are given below.

For the positive range of $\lambda$, the construction of the Curle tables was done using in effect only two solutions $u_o\xi^n$ and $u_o\sin\xi$ so that with two functions available for fitting purposes there was no fitting problem and a good agreement is to be expected. That with two extra solutions $u_o(1+\xi)$ and $u_o(\xi+\xi^3)$, to be fitted, the agreement is as good is encouraging. No values have been shown for high values of positive $\lambda$ as in this range, much extrapolation would have to be done to use the Curle tables and also in this particular range the fitting for the new tables was done really to the one solution $u_o\xi^n$. The argument for quoting functions for $\lambda \geqslant 0.09$ is that most problems having values of $\lambda$ in this range will in effect be of a Falkner-Skan, $u_o\xi^n$, type for which the values given should therefore be useful.

In the negative range of $\lambda$ with four solutions to fit, the Curle tables were produced as the result of a genuine fitting problem. In this region the mean modulus residual for L for the old tables is 0.0035 and for the new tables 0.0025 which is an improvement. For $l^2$ the corresponding mean modulus residual values are 0.00035 for the old tables and 0.0003 for the new tables. The close agreement may be traced to that fact that for $l^2$ in the original table the functions were fitted not by smooth curves, but given as a numerical tabulation which had been adjusted to give good agreement. The improvement for L shows that the linear form for $G_o(\lambda)$ which yielded the original tables has been improved upon.

Table 3.8 gives a comparison of the values of H using the new and old tables. In each case the values quoted are for values of $\lambda$ extending to approximately seventy percent of the separation value and for this range the errors are typically of order one percent of H. A direct comparison of these values near separation would produce misleading conclusions for the method. If $\Delta H$ denotes the difference between the exact value of H and the calculated value, and $\Delta\lambda$ denotes the difference between the values of $\lambda$ yielding these values of H then $\Delta H \approx \frac{dH}{d\lambda}\Delta\lambda$.

As is illustrated in Fig. 3.1 for the Falkner-Skan solution, near separation the gradient $dH/d\lambda$ is very steep and errors which are small in $\lambda$ can produce large errors in H.

Figure 3.2 shows the curve of H versus $\lambda$ at separation as given by the method using the new tables. At separation $l = 0$, yielding the relationships for the tabulated functions to be

$$0 = F_1(\lambda) - \mu G_1(\lambda)$$

$$F_o(\lambda) - \mu G_o(\lambda) = -2\lambda\left\{H + 2\right\}$$

which on elimination of $\mu$ gives

$$H = -\frac{1}{2\lambda}\left\{F_o(\lambda) - \frac{F_1(\lambda)}{G_1(\lambda)}G_o(\lambda)\right\} - 2$$

This curve is seen to be in good agreement with the known separation values which are indicated on the graph. The dotted curve represents the extrapolation of this curve into a region where there were no solutions available to tie down the tabulated functions.

# CHAPTER 4: The Approach for the Compressible Laminar Boundary Layer

## Section I: Introduction

Under the following conditions

a) zero heat transfer at the wall

b) the Prandtl number ($\sigma$) of the fluid is unity

c) the viscosity $\mu$ is assumed to be directly proportional to the absolute temperature T.

it has been shown independently by Illingworth[11] and Stewartson[19], that a compressible laminar boundary layer problem can be reduced exactly to an associated incompressible problem. This chapter deals with the extension of the Curle two parameter method to such compressible problems and deals in particular with the compressible boundary layer flow with external velocity $u_1 = u_\circ(1-\xi)$. $\xi = \frac{x}{c}$ .

In the next section an outline is given of the Stewartson-Illingworth transformation together with a brief description of the effect which compressibility has on the mainstream flow.

The third section deals with an idea due to Stewartson for assessing the relative accuracies of approximate methods when applied to a compressible boundary layer problem. Stewartson demonstrates his idea with respect to the methods of Howarth and Pohlhausen by applying these methods to the compressible flow with external velocity $u_1 = u_\circ(1-\xi)$. $\xi = \frac{x}{c}$ . Using both these methods the variation of the distance to separation, $x_s$, with Mach number for this problem is investigated. In each case the distance to separation, $x_s$. tends to zero as $M_\circ$ tends to infinity provided $\gamma$

is greater than a critical value, and these conditions on $\gamma$ are compared with the exact condition which may be obtained from the differential equation of the boundary layer. By virtue of its construction it is known that the method of Howarth will give the better answer for small Mach number. It is also found that the condition on $\gamma$ for this method is much closer to the condition for the exact solution than is that for Pohlhausen's method and so the Howarth method is also better than the Pohlhausen method for large Mach number. In this sense the condition on $\gamma$ may be used to determine the relative accuracies of approximate methods at high Mach number.

The fourth section deals with the extension of Stewartson's idea to the Curle two-parameter method in its application to the same compressible flow with external velocity $u_1 = u_o(1-\xi)$. An analysis similar to that for the methods of Howarth and Pohlhausen is carried out using both the original and the new values for the tabulated functions. It is found that the original values yield a condition on $\gamma$ in significantly better agreement with the exact condition than either of the results for the methods of Howarth and Pohlhausen, and that the new tables yield a $\gamma$ in still better agreement. These conditions are then used to explain certain features of the results, yielded by approximate methods, for the separation position for the compressible flow with external velocity $u_1 = u_o(1-\xi)$ at a leading edge Mach number, $M_o = 4$. The section is concluded with a calculation of the separation position for this problem using the Curle two parameter method with the new tables.

The chapter is concluded with a section on further approximations on the Stewartson transformation and conditions on $\gamma$ for more complicated external flows in the compressible plane to attain the separation solution on transformation as the Mach number in the compressible plane becomes infinite.

Section II: The Stewartson-Illingworth Transformation

As has been mentioned in the introduction to this chapter under the conditions of

a) zero heat transfer at the wall

b) the Prandtl number ($\sigma$) of the fluid equal to unity

c) the viscosity $\mu$ assumed to be directly proportional to the absolute temperature T,

a compressible laminar boundary layer problem can be reduced exactly to an associated incompressible problem.

The transformation is as follows. If we denote the velocity components, external velocity, stream function, etc, in the compressible plane by the normal symbols ie u, v, $u_1$, $\psi$ etc, and let a represent the velocity of sound in air, then we may write,

$$\rho u = \rho_o \frac{\partial \psi}{\partial y} \; : \; \rho v = -\rho_o \frac{\partial \psi}{\partial x}$$

satisfying the compressible form of the continuity equation automatically, and introduce the new independent variables X, Y, where

$$X = \int_o^x \left(\frac{a_1}{a_o}\right)^{\frac{3\gamma-1}{\gamma-1}} dx: \qquad Y = \frac{a_1}{a_o}\int_o^y \frac{\rho}{\rho_o}\, dy$$

where suffix 1 refers to local values at the edge of the boundary layer and suffix 0 to standard reference conditions at the leading edge. Then with

$$U = \frac{\partial \psi}{\partial Y}, \qquad V = -\frac{\partial \psi}{\partial X}$$

the continuity and momentum equations become simply $\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0$ and

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = U_1 \frac{dU_1}{dX} + \nu_o \frac{\partial^2 U}{\partial Y^2}$$ where $U_1 = (\frac{a_o}{a_1})\, u_1$ is the value of $U$ at the edge of the boundary layer.

The effect of the transformation may be seen as follows. From the Eulerian equations of motion in the main stream of the compressible flow it follows that

$$a_1^2 + \frac{\gamma-1}{2}\, u_1^2 = \text{const} = a_o^2 + \frac{\gamma-1}{2}\, u_o^2$$

$$(\text{Hence } a_1 \frac{da_1}{dx} = -\frac{(\gamma-1)}{2}\, u_1 \frac{du_1}{dx})$$

If $u_1$ is an increasing function of x then $a_1$ is decreasing and so $u_1/a_1$ increases more rapidly than $u_1$. Similarly if $u_1$ is decreasing, $u_1/a_1$ decreases more rapidly and so the effect of compressibility is to emphasise any change in the velocity. If $u_1$ is constant then so is $u_1/a_1$ and the only effect of compressibility is to change the scale of Y. Changes in kinematic viscosity as represented by the change in $\nu_o$ manifest themselves solely in the scale factor in the direction normal to the boundary: changes in compressibility as represented by changes in $a_o$, for example, enter at every stage, affecting the scale factors both along and normal to the boundary and the reference velocity distribution in the mainstream.

This analysis gives a guide as to the difficulties of applying approximate methods to compressible boundary layer problems, at higher

Mach numbers. It is seen that the Stewartson-Illingworth transformation is two-fold in its operation. Given an external velocity $u_1(x)$ in the physical (compressible) plane, the transformation yields an associated external velocity $U_1(X)$ in the transformed plane. Both the velocity and the co-ordinate along the wall are transformed and the severity of these transformations tends to increase with Mach number. It is not difficult to imagine the possibility that although $u_1(x)$ may be a simple analytic form, the associated $U_1(X)$ could, at higher Mach numbers, be almost singular in nature and so be quite different from any of the external velocities for which the better existing methods (e.g. Thwaites', Stratford's) have been found to give good results. In fact, for the compressible boundary layer problem with external velocity $u_1 = u_\circ(1-\xi)$, $\xi = \frac{x}{c}$ at a leading edge Mach number of four it has been shown that whereas the external velocity gradient $du_1/dx$ in the physical plane is everywhere equal to $-u_\circ/c$, in the transformed plane the associated velocity gradient $dU_1/dX$ decreases from roughly minus $3u_\circ/c$ at the leading edge, to minus $u_\circ/2c$ at about $X/c = 0.16$, corresponding to $x/c = 0.07$.

It therefore seems possible that, at high Mach number at any rate, velocity gradients in the associated incompressible flow may sometimes be of an extreme nature and existing calculation methods, based on solutions to problems having more moderate velocity gradients, may therefore be inadequate for such flows.

## Section III: An idea due to Stewartson for assessing the relative accuracies of approximate methods (when applied to certain compressible boundary layer problems)

This section deals with an idea due to Stewartson[19] for assessing the relative accuracies of approximate methods when applied to certain compressible boundary layer problems and is illustrated with respect to the methods of Howarth and Pohlhausen in their application to the compressible boundary layer problem with external velocity $u_1 = u_0(1 - \xi)$, $\xi = \frac{x}{c}$.

Each of the above methods reduces to the solution of an ordinary differential equation, which results from making some approximation to quantities related by the momentum integral equation. These equations are examined for $u_1 = u_0(1 - \xi)$ and from each a condition on $\gamma$, the ratio of the specific heats, is derived such that as the Mach number (at the leading edge) in the compressible plane tends to infinity, the separation position tends to zero. These conditions are then compared with the exact condition and it is found that the condition for the Howarth method, known by virtue of its construction to be the better method for this problem, is much closer to the exact condition than that of Pohlhausen's method. This condition is then regarded as a method of 'grading' the methods at high Mach numbers.

(a) The Method of Karman-Pohlhausen

(Notation)

(We use x instead of $\xi$ to agree with Howarth's notation and use small $u_1$, $u_0$ to represent quantities in the compressible plane rather than $U_1$, $U_o$ as used by Howarth.)

The equation for $\lambda$, the non-dimensional pressure gradient parameter, is

$$\frac{d\lambda}{dx} = \frac{1}{u_1}\frac{du_1}{dx}\left[ g(\lambda)\left\{ 1 + \frac{\gamma-1}{2} M_1^{\ 2}\right\} + (3\gamma-2)\, M_1^{\ 2}\left\{\lambda^2 h(\lambda) + \lambda\right\}\right] \qquad (4.3.1)$$

where $\quad M_1 = \frac{u_1}{a_1}$ and $g(\lambda) = \frac{15120 - 2784\lambda + 79\lambda^2 + 5/3\lambda^3}{(12-\lambda)(37 + (25/12)\lambda)}$

$$h(\lambda) = \frac{8 + 5/3\lambda}{(12-\lambda)(37 + (25/12)\lambda)}$$

This equation may be recast for $\lambda$ in terms of $M_1$ and $\gamma$ only as

$$M_1 \frac{d\lambda}{dM_1} = g(\lambda) + \frac{(3\gamma-2)M_1^{\ 2}}{1 + \frac{\gamma-1}{2} M_1^{\ 2}}\left[\lambda^2 h(\lambda) + \lambda\right] \qquad (4.3.2)$$

with initial condition $M_1 = \frac{u_o}{a_o} = M_o$ when $\lambda = 0$ and separation occurs at $\lambda$=

From consideration of the variations of $g(\lambda)$ and $h(\lambda)$ with $\lambda$ Stewartson then argues that if separation is to occur at all $M_1 \frac{d\lambda}{dM_1}$ must be positive when $\lambda \gg -12$ and since the value of $\lambda$ for which the right hand side is most likely to be zero is -12, the following condition is obtained setting $\lambda = -12$ there (set $\lambda = -12$ in 4.3.2 with $M_1 \frac{d\lambda}{dM_1} > 0$)

$$198 > \frac{18(3\gamma-2)M_1^{\ 2}}{1 + \frac{\gamma-1}{2} M_1^{\ 2}}$$

$$\text{or } 22 > M_1^{\ 2}\,(7 - 5\gamma) \qquad (4.3.3)$$

Separation must take place eventually, since for sufficiently large x, the mainstream velocity is reversed in direction. Hence if $\gamma < 1.4$ equation (4.3.3) must be regarded as an upper bound for the Mach number at separation ($M_s$) and hence $x_s$ has a positive non-zero lower bound.

In fact Stewartson shows that

$$x_s > 1 - \sqrt{\frac{11}{3\gamma-2}\left(\frac{\gamma-1}{2} + \frac{1}{M_o^2}\right)}$$

and therefore letting $M_o \to \infty$

$$x_s > 1 - \sqrt{\frac{11(\gamma-1)}{2(3\gamma-2)}}$$

Stewartson next considers the case $\gamma > 1.4$ and argues that if $\gamma > 1.4$ no matter how $M_1$ behaves between $x = 0$ and $x = x_s$) $\log \frac{M_1}{M_o}$ must be finite and therefore $x_s = 0\left(\frac{1}{M_o^2}\right)$ as $M_o \to \infty$. The critical value for $\gamma$ is 1.4, and since this value is commonly assumed for practical problems, it is discussed in more detail. With this value of $\gamma$ equation (4.3.2) takes on the form

$$(12-\lambda)\left(37 + \frac{25\lambda}{12}\right)M_1\left(1 + \frac{1}{5}M_1^2\right)\frac{d\lambda}{dM_1} = 15120-2784\lambda+79\lambda^2 + \frac{5}{3}\lambda^3 + \frac{7}{12}M_1^2\frac{\left[12+\lambda\right]^2}{(36-\lambda)} \qquad (4.3.4)$$

By integrating this equation near $\lambda = -12$ Stewartson states that the following results are produced

$$\log\frac{M_s}{M_o} = -\frac{\pi M_s}{10\sqrt{7}} + 0(1)$$

$$x_s = \frac{1}{3(10\log M_o)^2}\left[1+0(1)\right] \text{ as } M_o \to \infty$$

Collecting all the above results for this method together he presents the conclusions that the point of separations, $x_s$, tends to zero as $M_o \to \infty$ provided that $\gamma \geqslant 1.4$, while if $\gamma < 1.4$ this point has a non-zero lower bound, no matter how large $M_o$ may be.

(b) <u>The Method of Howarth</u>

In the next section of his paper Stewartson applies a similar type of analysis to the method of Howarth[10].

Howarth began by computing the velocity field in the boundary layer due to a linear main-stream velocity $u_1^* = b_o - b_1 x^*$, and tabulated the values of the skin friction and the momentum integral as functions of $\xi = b_1 x^*/b_o$. He then compared $U_1^*$ with $U_1$, the mainstream velocity under discussion, and determined the value of $\xi$ corresponding to z, (the independent variable of the $U_1$ flow). The principal assumption he made was that the momentum integral was continuous, and, finally he found that

$$\frac{d\xi}{dz} = -\frac{1}{U_1}\frac{dU_1}{dz}(1-\xi) + \frac{\mathcal{X}}{2d\mathcal{X}/d\xi} \cdot \frac{d^2U_1/dz^2}{dU_1/dz} \qquad (4.3.5)$$

where $\mathcal{X}$ was a known function of $\xi$ and $\mathcal{X}/d\mathcal{X}/d\xi$ was tabulated. Moreover, a table and a graph of $(\frac{\partial u}{\partial y})_o/U_1 \; (-\frac{dU_1}{dz})^{\frac{1}{2}}$ (4.3.6), a dimensionless function of $\xi$ were given in his paper. Hence having deduced $\xi$ in terms of x from the first-order differential equation (4.3.5), the skin friction may easily be deduced from (4.3.6). Separation occurs when $\xi = 0.120$.

Stewartson then points out that since it is possible to reduce any problem of compressible flow, satisfying the conditions of section 4.2, to a problem in incompressible flow, it follows that the process may be reversed and that the above method may be adapted to the solution of appropriate compressible flow problems.

Now the relationship between quantities in the compressible and incompressible planes are

$$U_1 = \frac{a_\circ}{a_1} u_1 = a_\circ M_1 \text{ where } M_1 = \frac{u_1}{a_1}$$

$$\frac{dU_1}{U_1} = \frac{du_1}{u_1} \left(1 + \frac{\gamma-1}{2} M_1^2\right)$$

and $$d\left(\log \frac{dU_1}{dz}\right) = d\left(\log \frac{du_1}{dx}\right) + (3\gamma - 2) M_1^2 \, d \log u_1$$

so that the differential equation becomes

$$\frac{d\xi}{dx} = \frac{\chi}{2}\frac{d\xi}{d\chi} \cdot \frac{d^2u_1/dx^2}{du_1/dx} - \frac{du_1/dx}{u_1}\left[(1-\xi)\left(1 + \frac{\gamma-1}{2} M_1^2\right) - \frac{(3\gamma-2)}{2}\chi \frac{d\xi}{d\chi} M_1^2\right] \tag{4.3.7}$$

In this form Howarth's method is now applicable to compressible boundary layers with a retarded mainstream. For the compressible flow problem $u_1 = u_\circ(1-x)$ the equation reduces to

$$-M_1 \frac{d\xi}{dM_1} = 1 - \xi - \frac{(3\gamma-2)\chi}{2d\chi/d\xi} \cdot \frac{M_1^2}{1 + \frac{\gamma-1}{2} M_1^2} \tag{4.3.8}$$

and the integration of this first-order differential equation may easily be carried out. The initial condition is that $M = M_\circ$ when $\xi = 0$ and separation occurs when $\xi = 0.120$. This equation (4.3.8) is closely analogous to the corresponding differential equation obtained in Karman-Pohlhausen's method. Stewartson then compares the solutions by the method of Howarth and of Karman Pohlhausen for problems at various leading edge Mach numbers. He produces a table of the positions of separation point for various values of $M_\circ$ as given by the methods of Howarth and of Karman Pohlhausen. From this table it is seen that

Pohlhausen's method asserts that the point of separation occurs considerably later than does Howarth's method. From comparisons with known solutions it appears that Howarth's method would overestimate the skin friction if $d^2u_1/dz^2$ were positive. For the problem under consideration Stewartson points out that this is true and Howarth's method should overestimate the point of separation and is therefore a closer approximation. He goes on to determine the condition on $\gamma$ for the point of separation to tend to the leading edge of the plate as the Mach number $M_o$ tends to infinity. The condition is that the right hand side of equation (4.3.8) should never vanish. As with the Karman-Pohlhausen method the most probable place for this to occur is where $\xi = 0.120$, and hence

$$0.880 > \frac{3\gamma-2}{2}(0.151)\frac{M_1^2}{1+\frac{\gamma-1}{2}M_1^2}$$

or

$$1.760 > M_1^2\,(0.578 - 0.427\gamma)$$

so that if the point of separation is to be as required

$$\gamma \geq 1.35(4)$$

The equality sign may be dealt with as in the Pohlhausen method and the critical value compared with 1.4 obtained there. If $\gamma < 1.35(4)$, Howarth's method asserts that the point of separation has a positive non-zero lower bound. Stewartson argues that if it can be shown from the differential equation that the critical value of $\gamma$ is less than 1.35(4) then it can be asserted that Howarth's method is better than Pohlhausen's.

(c) An exact solution with $M_o \gg 1$

The argument Stewartson uses is as follows. It is supposed that $M_o$ is very large and $t = xM_o^2$, then if t is large, but definite so that $t/M_o^2$ is small then

$$a_1^2 = a_o^2\,(1 + (\gamma - 1)t)$$

and
$$u_1 = a_o M_o + O(\tfrac{1}{M_o})$$

Hence
$$M_o^2\, z = \int_0^t (1 + (\gamma-1)t_1)^{(3\gamma-1)/2(\gamma-1)}\, dt_1$$

so that
$$z \sim \left[1 + (\gamma-1)t\right]^{(5\gamma-3)/2(\gamma-1)}$$

(where $f \sim g$ is to be interpreted as $f = Ag\left[1 + o(1)\right]$ for large t, where A is a positive constant). Hence

$$U_1 = a_o\, M_1 \sim z^{-(\gamma-1)/(5\gamma-3)}$$

Now the solution of this problem is known, having been studied by Hartree[8]. He found that if $U_1 = Az^{-m}$ then $(\psi_{yy})_{y=0} = z^{-\frac{1}{2}(3m+1)}\, f(m)$

(4.2.9)

where f(m) is some function of m.

Stewartson then applies this result to his problem. If $(\psi_{yy})_{y=0} > 0$ for large t, then, since $U_1$ is a strictly decreasing function $(\psi_{yy})_{y=0}$ is also strictly decreasing. $(\psi_{yy})_{y=0} > 0$ for all finite t, and hence separation occurs for some infinite value of t. Conversely, if $(\psi_{yy})_{y=0} < 0$ for large t, then separation has occurred for some finite t. Stewartson points out that the whole theory breaks down at separation and that no deduction is possible directly, but indicates that the argument can proceed as follows. He supposes that separation does not occur for some finite t and $(\psi_{yy})_{y=0} < 0$ for large t. Then since $(\psi_{yy})_{y=0} > 0$ when $t = 0$ there is a point at which $(\psi_{yy})_{y=0} = 0$, i.e. a point of separation, and the hypothesis breaks down. From (4.2.9) the critical value of m is given by $f(m) = 0$, or $m = 0.090428(53)$ using Evans' more recent results (6) for this problem.

In terms of $\gamma$ the critical value is $\gamma = 1.3301(2)$. Hence if $\gamma > 1.3301(2)$ the point of separation occurs for a finite $t = t_s$, i.e. when $x_s = t_s/M_0^2$, and approaches the origin as $M_0 \to \infty$. If $\gamma < 1.3301(2)$ the point of separation occurs at an infinite value of t. Hence from the conclusions of (a) and (b), in terms of x it probably occurs at a non-zero distance from the leading edge.

As has been demonstrated the method of Howarth produces better results as the Mach number increases and also a condition on $\gamma$ closer to the exact condition. On the basis of this it seems reasonable to use this condition as a means of assessing the relative accuracies of approximate methods.

Section IV: The Extension of Stewartson's analysis to the Curle two parameter method with special reference to its application to the compressible flow with external velocity $u_1 = u_0(1-\xi)$

4.4.1 In this section we show how Stewartson's analysis applied in the previous section to the methods of Howarth and Pohlhausen for the compressible flow problem with external velocity distribution $u_1 = u_0(1-\xi)$ may be extended to the Curle two parameter method in its application to the same problem. From the Stewartson transformation we have that, as the Mach number at the leading edge in the compressible plane tends to infinity, the associated external flow in the incompressible plane should tend to that for the Falkner Skan separation solution.

For the Curle two parameter method this requirement can be expressed by a relationship between the parameters $\Lambda$ and $\Pi$ (those associated with the transformed flow) from which by use of the tabulated functions used in the method the critical value of $\gamma$ can be worked out. Use of the original tables yields a value of $\gamma$ in significantly better agreement with the exact condition than either of the conditions yielded by the methods of Howarth and Pohlhausen and use of the new tables produces a yet better agreement.

The compressible flow with external velocity $u_1 = u_0(1-\xi)$ at a leading edge Mach number of 4 is then examined to verify this interpretation on the critical value of $\gamma$ as a guide to the accuracy of the method. This particular problem was chosen as it is one for which a reliable and highly accurate solution has been computed, by P. G. Williams of University College London, to be $\xi_{sep} = 0.056(4)$. The corresponding results from the 2-parameter method were $\xi_{sep} = 0.054(7)$ using the old tables and $\xi_{sep} = 0.055(1)$ using the new tables. The flow $u_1 = u_0(1-\xi)$ was then examined for further values of the leading edge Mach number.

Stewartson has shown that provided $\gamma \geqslant 1.33$, the compressible external flow $u_1 = u_0(1-\xi)$ transforms into a velocity profile which attains the Falkner-Skan separation solution profile as the leading edge Mach number becomes infinite. The results yielded by the calculations for high Mach numbers with $\gamma = 1.4$ gave separation values of the parameters $\Lambda$ and $\mu$ which were consistent with Stewartson's arguments.

4.4.2 Theory

Denote by $a_o$, $a_1$, u, v, x, y, $u_1$ the usual quantities in the compressible plane and by $U_1$, X, Y, U, V the usual quantities in the associated incompressible plane where

$$U_1 = \left(\frac{a_o}{a_1}\right) u_1 , \quad X = \int_o^x \left(\frac{a_1}{a_o}\right)^{\frac{3\gamma-1}{\gamma-1}} dx; \quad Y = \frac{a_1}{a_o}\int_o^y \frac{\rho_o}{\rho_o}\, dy \ .$$

If we also denote the pressure gradient parameter in the compressible plane by $\Lambda$ then the momentum integral equation is

$$\frac{d}{dX}\left(\frac{\Lambda}{U_1'}\right) = \frac{L}{U_1} \qquad (4.4.1)$$

$$U_1\left\{\frac{1}{U_1'}\frac{d\Lambda}{dX} - \Lambda\frac{U_1''}{(U_1')^2}\right\} = L = F_o(\Lambda) - \mu\, G_o(\Lambda) \qquad (4.4.2)$$

$$= F_o(\Lambda) - \Lambda^2 \frac{U_1 U_1''}{(U_1')^2} G_o(\Lambda) \qquad (4.4.3)$$

The above equation may be rewritten in the form

$$\frac{U_1}{U_1'}\ \frac{d\Lambda}{dX} = F_o(\Lambda) + \left\{\Lambda - \Lambda^2\, G_o(\Lambda)\right\} U_1 U_1''/(U_1')^2 \qquad (4.4.4)$$

This may be further rewritten as follows

$$U_1 = \left(\frac{a_o}{a_1}\right) u_1 \qquad (4.4.5)$$

where
$$\left(\frac{a_1}{a_o}\right)^2 = 1 + \tfrac{1}{2}(\gamma-1)\, M_o^{\,2}\left(1 - \frac{u_1^{\,2}}{u_o^{\,2}}\right) \qquad (4.4.6)$$

$$= \frac{1 + \frac{1}{2}(\gamma-1)M_o^{\,2}}{1 + \frac{1}{2}(\gamma-1)M_1^{\,2}} \qquad (4.4.7)$$

$$M_1 = \frac{u_1}{a_1} = \frac{U_1}{a_\circ} = \frac{M_\circ}{u_\circ} U_1 \tag{4.4.8}$$

$$\therefore \quad \frac{dM_1}{dX} = \frac{M_\circ}{u_\circ}\frac{dU_1}{dX} = \frac{M_1}{u_1} U_1' \tag{4.4.9}$$

$$\therefore \quad \frac{U_1}{U_1'}\frac{d\Lambda}{dX} = \frac{U_1}{U_1'}\frac{d\Lambda}{dM_1}\frac{dM_1}{dX} = M_1 \frac{d\Lambda}{dM_1} \tag{4.4.10}$$

$\therefore$ Equation (4) becomes

$$M_1 \frac{d\Lambda}{dM_1} = F_\circ(\Lambda) + \left\{\Lambda - \Lambda^2 G_\circ(\Lambda)\right\} U_1 U_1''/(U_1')^2 \tag{4.4.11}$$

In general we have

$$\frac{dU_1}{dX} = \left(\frac{a_\circ}{a_1}\right)^{\frac{4\gamma-2}{\gamma-1}} \frac{du_1}{dx}\left(1 + \frac{\gamma-1}{2}\frac{M_\circ^2}{u_\circ^2} u_1^2 \left(\frac{a_\circ}{a_1}\right)^2\right) \tag{4.4.12}$$

$$\frac{d^2U_1}{dX^2} = \left(\frac{a_\circ}{a_1}\right)^{\frac{7\gamma-3}{\gamma-1}} \left(1 + \frac{\gamma-1}{2}\frac{M_\circ^2}{u_\circ^2}\left(\frac{a_\circ}{a_1}\right)^2 u_1^2\right)\left(\frac{d^2u_1}{dx^2} + (3\gamma-2)\frac{M_\circ^2}{u_\circ^2}\left(\frac{du_1}{dx}\right)^2\left(\frac{a_\circ}{a_1}\right)^2\right) \tag{4.4.13}$$

so that

$$\frac{U_1U_1''}{(U_1')^2} = u_1 \frac{\left[\frac{d^2u_1}{dx^2} + (3\gamma-2)\frac{M_\circ^2}{u_\circ^2} u_1 \left(\frac{du_1}{dx}\right)^2\left(\frac{a_\circ}{a_1}\right)^2\right]}{\left(\frac{du_1}{dx}\right)^2\left[1 + \frac{\gamma-1}{2}\frac{M_\circ^2}{u_\circ^2} u_1^2 \left(\frac{a_\circ}{a_1}\right)^2\right]} \tag{4.4.14}$$

Now for the particular flow $u_1 = u_\circ(1-\xi)$; $\frac{d^2u_1}{dx^2} \equiv 0$ so, after some cancellation, we get

$$\frac{U_1U_1''}{(U_1')^2} = \frac{(3\gamma-2)\, M_\circ^2\, u_1^2/u_\circ^2}{\left(\frac{a_1}{a_\circ}\right)^2 + \frac{(\gamma-1)}{2}\frac{M_\circ^2}{u_\circ^2} u_1^2} \tag{4.4.15}$$

$$= \frac{(3\gamma-2)\ M_o^2\ u_1^2/u_o^2}{1+\frac{1}{2}\ (\gamma-1)\ M_o^2} \tag{4.4.16}$$

using equation (6)

Also since $M_1 = \frac{u_1}{a_1}$ ; $M_o = \frac{u_o}{a_o}$

$$\left(\frac{a_1}{a_o}\right)^2 = \left(\frac{u_1}{u_o}\right)^2 \left(\frac{M_o}{M_1}\right)^2 \tag{4.4.17}$$

so, using (7)

$$\left(\frac{u_1}{u_o}\right)^2 = \left(\frac{M_1}{M_o}\right)^2 \cdot \frac{1+\frac{\gamma-1}{2}\ M_o^2}{1+\frac{\gamma-1}{2}\ M_1^2} \tag{4.4.18}$$

Substitution of (18) into (16) yields

$$\frac{U_1 U_1''}{(U_1')^2} = \frac{(3\gamma-2)\ M_1^2}{1+\frac{\gamma-1}{2}\ M_1^2} \tag{4.4.19}$$

so that equation (11) becomes

$$M_1\ \frac{d\Lambda}{dM_1} = F_o(\Lambda) + \left[\Lambda - \Lambda^2\ G_o(\Lambda)\right] \frac{(3\gamma-2)\ M_1^2}{1+\frac{\gamma-1}{2}\ M_1^2} \tag{4.4.20}$$

which is of the same form as the equation (5.2) given by Stewartson. As $M_1 \to \infty$, equation (20) becomes

$$M_1\ \frac{d\Lambda}{dM_1} \simeq F_o(\Lambda) + k\left[\Lambda - \Lambda^2\ G_o(\Lambda)\right] \tag{4.4.21}$$

$$\text{where } k = \frac{2(3\gamma-2)}{\gamma-1}$$

Now, following the arguments of Stewartson[19], we require that the right hand side of 4.4.21 should become zero only at separation. This gives us a separation condition of

$$F_0(\Lambda) + k\left[\Lambda - \Lambda^2 G_0(\Lambda)\right] = 0 \qquad (4.4.22)$$

But the Curle method predicts separation where

$$F_1(\Lambda) - \mu G_1(\Lambda) = 0 \qquad (4.4.23)$$

and since $\mu = U_1 U_1''/(U_1')^2 \Lambda^2$

$$= k\Lambda^2 \qquad (4.4.24)$$

(4.4.23) becomes $$F_1(\Lambda) - k\Lambda^2 G_1(\Lambda) = 0 \qquad (4.4.25)$$

We therefore require to find k such that (4.4.22) and (4.4.25) are satisfied simultaneously. Elimination of k between (4.4.22) and (4.4.25) yields

$$\Lambda F_0 G_1 + F_1 (1 - \Lambda G_0) = 0 \qquad (4.3.26)$$

A method for determining k and hence the critical value of $\gamma$ is given below.

### 4.4.3 Calculation of the limiting condition on $\gamma$ for the original tables

The original tables yield

| $-\Lambda$ | $F_0(\Lambda)$ | $G_0(\Lambda)$ | $F_1(\Lambda)$ | $G_1(\Lambda)$ |
|---|---|---|---|---|
| 0.06 | 0.8053 | 0.48 | 0.0095 | 0.0971 |
| 0.07 | 0.8729 | 0.45 | 0.0047 | 0.0852 |
| 0.08 | 0.9434 | 0.42 | 0.0010 | 0.0728 |

which gives

| $+\Lambda$ | $\Lambda F_0 G_1 + F_1(1-\Lambda G_0)$ |
|---|---|
| -0.06 | 0.0050819 |
| -0.07 | -0.0003579 |
| -0.08 | -0.0044608 |

Now working in terms of rescaled variables we obtain the table

| $\Lambda$ | x | y | $k\Lambda^2$ | k |
|---|---|---|---|---|
| -0.06 | -1 | 0.50819 | 0.09784 | 27.178 |
| -0.07 | 0 | -0.03579 | 0.05516 | 11.257 |
| -0.08 | 1 | -0.44608 | 0.01374 | 2.147 |

To find the value of k when y = 0, ie when the two conditions are satisfied simultaneously, fit a quadratic in y through the values of k $\therefore\ k = a + by + cy^2$

$$
\begin{aligned}
27.178 &= a + 0.50819b + (0.50819)^2c \\
11.257 &= a - 0.03579b + (0.03579)^2c \\
2.147 &= a - 0.44608b + (0.44608)^2c
\end{aligned}
$$

or

$$
\begin{aligned}
53.480 &= 1.9678a + b + 0.50819c \\
314.530 &= 27.9408a - b + 0.03579c \\
4.813 &= 2.2418a - b + 0.44608c
\end{aligned}
$$

The solution of these equations yields a = 12.16988.

Therefore when y = 0, k = 12.16988, which in turn gives

$$\frac{2(3\gamma-2)}{\gamma-1} = 12.16988$$

$$\text{or } \gamma = 1.324(2) \qquad\qquad (\ c(c))$$

The corresponding condition for the Howarth method was $\gamma = 1.35(4)$ and for the Karman-Pohlhausen method $\gamma = 1.4$. Since the exact condition is $\gamma = 1.3301(2)$ it is seen that the condition for the Curle method is in significantly better agreement with the exact condition than either of the other two.

4.4.4 <u>Calculation of the limiting condition on $\gamma$ for the new tables</u>

The new tables yield

| $\Lambda$ | $F_0(\Lambda)$ | $G_0(\Lambda)$ | $F_1(\Lambda)$ | $G_1(\Lambda)$ |
|---|---|---|---|---|
| -0.060 | 0.80319 | 0.53677 | 0.00964 | 0.10059 |
| -0.070 | 0.87197 | 0.48496 | 0.00497 | 0.08903 |
| -0.080 | 0.94603 | 0.41391 | 0.00096 | 0.07666 |

which gives

| $\Lambda$ | $\Lambda F_\circ G_1 + F_1(1-\Lambda G_\circ)$ |
|---|---|
| -0.060 | 0.0051029 |
| -0.070 | -0.0002955 |
| -0.080 | -0.0048100 |

Now working in terms of rescaled variables we obtain the table

| $\Lambda$ | x | y | $k\Lambda^2$ | k |
|---|---|---|---|---|
| -0.060 | -1 | 0.51029 | 0.095834 | 26.6205 |
| -0.070 | 0 | -0.02955 | 0.055824 | 11.3926 |
| -0.080 | 1 | -0.48100 | 0.012523 | 1.95672 |

To find the value of k when y = 0, ie when the two conditions are satisfied simultaneously, fit a quadratic in y through the values of k. $\quad k = a + by + cy^2$

$$26.6205 = a + (0.51029)b + (0.51029)^2c$$
$$11.3926 = a + (-0.02955)b + (-0.02955)^2c$$
$$1.95672 = a + (-0.48100)b + (-0.48100)^2c$$

or

$$52.167395 = 1.959670a + b + 0.51029c$$
$$385.53637 = 33.840947a - b + 0.02955c$$
$$4.068025 = 2.079002a - b + 0.48100c$$

The solution of these equations yields a = 12.10339. Therefore when y = 0, the value of k is now k = 12.10339 which in turn gives

$$2(3\gamma-2)/\gamma-1 = 12.10339$$

$$\text{or } \gamma = 1.328$$

which is in even better agreement with the exact condition.

4.4.5 Application of the method to a compressible boundary layer problem

The method was then used to predict the distance to separation for the compressible flow with external velocity distribution $u_1 = u_\circ(1-\xi)$ at a leading edge Mach number of 4, where the Prandtl number is unity, there is zero heat transfer and the viscosity is proportional to the absolute temperature. This particular problem was chosen for investigation as a highly accurate estimate of the distance to separation has been computed by P. G. Williams of University College, London, to be $\xi_{sep} = 0.056(4)$.

By using the Stewartson-Illingworth transformation the problem is first transformed into an associated incompressible problem. The parameters $\Lambda$ and $\mu$ for this incompressible problem are given in terms of the compressible flow variables by the following relationships

$$\Lambda = 0.45 \frac{du_1^*}{d\xi} T^{-5/2} \left(1+\frac{\gamma-1}{2} M_o^2\right)(u_1^*)^{-6} \left\{ \int_o^\xi (1+2.22g(\Lambda,\mu)) \, u_1^{*5} T^{3/2} d\xi \right\} \quad (a)$$

$$\mu = \frac{\Lambda^2 u_1^* \left[ \frac{d^2u_1^*}{d\xi^2}(T) + (3\gamma-2)u_1^* \left(\frac{du_1^*}{d\xi}\right)^2 M_o^2 \right]}{\left(\frac{du_1^*}{d\xi}\right)^2 \left(1+\frac{\gamma-1}{2} M_o^2\right)} \quad (b)$$

where $u_1^* = u_1/u_o$ , $\frac{du_1^*}{d\xi}$ and $\frac{d^2u_1^*}{d\xi^2}$ refer to the compressible plane and $T = 1 + \frac{\gamma-1}{2} M_o^2 (1 - u_1^{*2})$.

The application of the method then involves the following iterative routine. The function $g(\Lambda\ \mu)$ is initially set equal to zero and by using relationships (a) and (b) a table of values of $\Lambda$ and $\mu$ for a range of values of $\xi$ is constructed. The function $l^2 = F_1(\Lambda) - \mu G_1(\Lambda)$ may then be evaluated by use of the tabulated functions, interpolating where necessary, for each value of $\xi$. Since separation is taken to occur where $l^2 = 0$, by interpolation on this table the value of $\xi_{sep}$

may be obtained. By using the tabulated functions $F_o(\Lambda)$ and $G_o(\Lambda)$ and the relationship $g(\Lambda,\mu) = F_o(\Lambda) - 0.45 + 6\Lambda - \mu G_o(\Lambda)$, for each value of $\xi$ a new value of $g(\Lambda,\mu)$ may be calculated. With these values of $g(\Lambda,\mu)$ the relationships (a) and (b) may then be evaluated again to yield new values of $\Lambda$ and $\mu$ from which by use of the tables a new estimate of the distance to separation, $\xi_{sep}$, is obtained. As above a new tabulation of $g(\Lambda,\mu)$ may be obtained and the iteration process continued on until two successive estimates of the distance to separation have converged to within some acceptable tolerance.

In practice a computer program was written to perform these calculations. A steplength of 0.001 was used for $\xi$ and the process was allowed to continue until two successive estimates of the distance to separation differed by less than 0.00001. For the problem under consideration this was achieved after four iterations which on an Elliot 4100 computer took approximately a minute. The results obtained were $\xi_{sep} = 0.05474$ for the old tables and $\xi_{sep} = 0.05506$ for the new tables.

The computer program was then slightly modified to yield the values of $\Lambda$ and $\mu$ at separation corresponding to the external flow $u_1 = u_o(1-\xi)$ at various leading edge Mach numbers. Using the new tables, the following results were obtained

| $M_o$ | $\xi_{sep}$ | $\Lambda_{sep}$ | $\mu_{sep}$ |
|---|---|---|---|
| 0 | 0.11843 | -0.083(0) | 0.000(0) |
| 2 | 0.08828 | -0.077(5) | 0.024(4) |
| 4 | 0.05506 | -0.073(8) | 0.040(8) |
| 6 | 0.03596 | -0.072(3) | 0.046(9) |
| 8 | 0.02491 | -0.071(6) | 0.049(7) |
| 10 | 0.01809 | -0.071(2) | 0.051(2) |

Now Stewartson has shown that provided $\gamma \geq 1.5501$(2) this compressible external flow $u_1 = u_0(1-\xi)$ will transform into a velocity distribution which tends to the Falkner-Skan separation profiles as the leading edge Mach number becomes infinite. For this particular profile the exact values of the parameters $\Lambda$ and $\mu$ are -0.0681 and 0.0560 respectively and by interpolation from the tabulated functions in the separation condition $F_1(\Lambda) - \mu G_1(\Lambda) = 0$ the corresponding values indicated by the method are -0.0695 and 0.0583 respectively. The values of $\Lambda$ and $\mu$ given above were obtained through the computer program by linear interpolation using the tabulated functions, and it does not seem that extrapolation to infinite Mach number from these given values will produce the indicated limiting values. With such values linear interpolation may well be inadequate and a modification to quadratic interpolation may produce values which do extrapolate to the expected limiting values.

One feature which the above table does illustrate is the adjustment of the values of $\Lambda$ and $\mu$ at separation, as against the assumption of methods, such as Thwaites', which predict separation to occur for a fixed value of $\Lambda$.

## V. The extension of Stewartson's Analysis to further compressible flows

In the previous sections Stewartson's analysis has been presented for the incompressible flow associated through the Stewartson-Illingworth transformation with the compressible external flow $u_1 = u_0(1-\xi)$, to tend to the Falkner-Skan separation solution as the leading edge Mach number in the compressible plane tended to infinity. This was found to reduce to the condition $\gamma \geq 1.3301(2)$ a condition which would be satisfied in most experimental situations. It was also found that this condition could be used to assess the relative accuracies of approximate methods in tackling the above compressible flow problem. It is therefore interesting to see what further restrictions may be put on the transform, and what conditions these yield on $\gamma$, that more general flows in the compressible plane should transform into flows which attain the Falkner Skan separation solution as the leading edge Mach number becomes infinite.

We therefore consider two types of external flow in the compressible plane.

1) $u_1 = u_\circ(1 - \alpha\xi^n)$ , $\alpha > 0$

2) $u_1 = \frac{n}{n-1} u_\circ(\xi - \frac{1}{n}\xi^n)$; $u_1 = u_\circ$ when $\xi = 1$, $|u_1| \leq u_\circ$.

Case (1). The analysis for $u_1 = u_\circ(1 - \alpha\xi^n)$

Let $a_\circ$ = velocity of sound where $\xi = 0$: $a_1$ = velocity of sound in mainstream.

$M_\circ$ - Mach number where $x = 0$: $M_1$ = Mach number in mainstream.

We have by Bernoulli.

$$a_\circ^2 + \frac{\gamma-1}{2} u_\circ^2 = a_1^2 + \frac{\gamma-1}{2} u_1^2$$

$$\therefore \left(\frac{a_1}{a_\circ}\right)^2 = 1 + M_\circ^2 \frac{(\gamma-1)}{2}\left(1 - \frac{u_1^2}{u_\circ^2}\right)$$

When $u_1 = u_\circ(1 - \alpha\xi^n)$

$$1 - \frac{u_1^2}{u_\circ^2} = 1 - (1 - \alpha\xi^n)^2 = \alpha\xi^n(2 - \alpha\xi^n)$$

$$\therefore \left(\frac{a_1}{a_\circ}\right)^2 = 1 + M_\circ^2 \frac{(\gamma-1)}{2} \alpha\xi^n (2 - \alpha\xi^n)$$

Now define $t = M_\circ^{2/n}\,\xi$ where $M_\circ$ is very large, t large but definite and $t/M_\circ^{2/n}$ very small.

$\therefore$ For large t, $\left(\frac{a_1}{a_\circ}\right)^2 \sim 1 + \alpha(\gamma-1)t^n \sim ct^n$

$$\therefore \quad X = \int_\circ^\xi \left(\frac{a_1}{a_\circ}\right)^{\frac{3\gamma-1}{\gamma-1}} d\xi \sim \frac{1}{M_\circ^{2/n}} \int_\circ^t t_1^{\,n/2\left(\frac{3\gamma-1}{\gamma-1}\right)} dt_1$$

$$\sim Bt^{\frac{(3n+2)\gamma-(n+2)}{2(\gamma-1)}}$$

$$\text{and } U_1 = \left(\frac{a_\circ}{a_1}\right) u_1 \sim t^{-n/2}\, u_\circ\left(1 - \alpha\left(\frac{t}{M_\circ^{2/n}}\right)^n\right)$$

$$\sim u_\circ t^{-n/2} \text{ since } t/M_\circ^{2/n} \text{ is small}$$

$$\sim u_\circ X^{-\beta(\gamma)}$$

$$\text{where } \beta(\gamma) = \frac{n(\gamma-1)}{(3n+2)\gamma-(n+2)}$$

For this velocity profile to attain the Falkner-Skan separation profile $U_1 = AX^{-m}$ where $m = 0.0904$ we require $\beta(\gamma) \geqslant 0.0904$ ie

$$\frac{n(\gamma-1)}{(3n+2)\gamma-(n+2)} \geqslant 0.0904$$

which reduces to $$\gamma \geqslant \frac{0.9096n - 0.1808}{0.7288n - 0.1808}$$

Case (2). The analysis for $u_1 = \frac{n}{n-1} u_\circ\left(\xi - \frac{\xi^n}{n}\right)$; $u_1 = u_\circ$ when $\xi = 1$, $|u_1| \leqslant u_\circ$

Let $a_\circ$ = velocity of sound when $\xi = 1$; $a_1$ = velocity of sound in mainstream.

$M_\circ$ = Mach number when $\xi = 1$, $M_1$ = Mach number in mainstream.

Define a new variable $\xi_1$, measured from the pressure minimum by

$\xi = 1 + \xi_1$.

In terms of this

$$u_1 = \frac{n}{n-1} u_\circ (1 + \xi_1 - \frac{1}{n} (1 + \xi_1)^n)$$

$$= \frac{n}{n-1} u_\circ (1 + \xi_1 - \frac{1}{n} (1 + n\xi_1 + {}^nC_2 \xi_1^2 + \ldots \text{ for small } \xi_1$$

$$\approx u_\circ (1 - {}^nC_2 \frac{1}{n-1} \xi_1^2 + \ldots \text{ higher order terms.}$$

Now define $X_1 = \int_1^{\xi_1} (\frac{a_1}{a_\circ})^{\frac{3\gamma-1}{\gamma-1}} d\xi_1$; $U_1 = (\frac{a_\circ}{a_1}) u_1$

Now $(\frac{a_1}{a_\circ})^2 = 1 + \frac{\gamma-1}{2} M_\circ^2 (1 - \frac{u_1^2}{u_\circ^2}) = 1 + \frac{\gamma-1}{2} M_\circ^2 \alpha \xi_1^2 (2 - \alpha\xi_1^2)$ where $\alpha = \frac{{}^nC_2}{n-1}$

On defining $t = M_\circ \xi_1$ where t is large but definite, $\xi_1$ small and $M_\circ$ very large so that $t/M_\circ$ small we have

$$(\frac{a_1}{a_\circ})^2 \sim 1 + \alpha(\gamma-1)t^2 \sim \alpha^* t^2 \text{ for large } t$$

$$\therefore X_1 = \int_\circ^t t_1^{\frac{3\gamma-1}{\gamma-1}} dt_1 \sim t^{\frac{4\gamma-2}{\gamma-1}}$$

But $U_1 = (\frac{a_\circ}{a_1}) u_1 \sim t^{-1} u_\circ (1 - \alpha(\frac{t}{M_\circ})^2) \sim u_\circ t^{-1}$

$$\sim A X_1^{-((\gamma-1)/(4\gamma-2))}$$

For this transformed flow to attain the Falkner-Skan separation profile we require $\frac{\gamma-1}{4\ -2} \gg 0.0904$ ie $\gamma \gg 1.283$.

In this short section an attempt has been made to suggest how Stewartson's analysis for the compressible boundary layer problem $u_1 = u_\circ(1-\xi)$ may be extended to further compressible boundary layer problems. Two approximations have been given and the corresponding conditions on $\gamma$ produced, which may prove useful in the investigation of further compressible boundary layer problems.

## Concluding Remarks

In this chapter, some of the difficulties in tackling compressible laminar boundary layer problems have been investigated with special reference to the compressible flow problem with external velocity distribution $u_1 = u_o(1-\xi)$. An account has been given of Stewartson's method for assessing the relative accuracies of approximate methods in their application to this particular problem and the analysis has been extended to the Curle two-parameter method. The result produced for this method indicated that it should be highly accurate at higher Mach numbers and this was verified by the calculation of the separation position for this flow at a leading edge Mach number of 4. The possibility of extending Stewartson's analysis to further boundary layer problems has also been investigated and two possible approximations have been investigated.

# CHAPTER 5: Concluding Remarks

The thread running through this thesis has been the investigation of the Curle two-parameter method and in particular the tabulated functions which this method involves. In chapter 1 certain suggestions were given whereby these tables might be improved. In this concluding chapter the work which these suggestions generated, and the results which this work produced are reviewed and summarised.

At the outset there were two broad areas where it was thought that further investigation might prove profitable. One was in extending the range of $\lambda$ over which the method could be applied with some reliability. The other was in improving the existing tables by a better fitting of the data, and in the region of small $\lambda$ producing an analytic form for H.

As was mentioned in chapter 1, the original tabulated functions had been derived using only one solution, $u = u_0\xi^n$, for the range $\lambda \geqslant 0.0685$ which meant that the reliability of these tables was in doubt for this region. To amend this, two new solutions which yielded values of $\lambda$ in this region were tackled, namely $u_1 = u_0(1+\xi)$ and $u_1 = u_0(\xi+\xi^3)$. The approaches attempted for these were given in Chapter 2. Initially expansions (for small values of $\xi$) were attempted by Howarth-Blasius type series in terms of the non-dimensional coordinate $\xi$ for the skin friction, displacement thickness and momentum thickness. It was hoped that the required set of parameters $\left\{\lambda,\ \mu,\ H,\ L \text{ and } l\right\}$ could then be evaluated in terms of these. In practice this approach yielded only the initial development for very small values of $\xi$ and was limited by the poor convergence of the expansions involved.

For large values of $\xi$ it was assumed that the flow $u_1 = u_o(\xi + \xi^3)$ would behave as $u_1 \approx u_o \xi^3$ and an analysis was attempted to see what forms would be possible for the stream function in this case. The resulting analysis made interesting use of the work of Libby and Chen in their treatment of perturbation solutions of the Falkner-Skan equation.

To obtain the eventual parameters what was done was to express the parameters $\lambda$, H and l in terms of the co-ordinate $\xi$, and by elimination of $\xi$, produce H and l as functions of $\lambda$. The function L was produced by use of the relationship $L = 2\left\{l - \lambda(H+2)\right\}$. This approach produced very good results and only very small corrections were necessary to attain the correct limiting values in both the case of the flow $u_1 = u_o(1+\xi)$ and $u_1 = u_o(\xi+\xi^3)$.

The second chapter also contained some examples of the use of Euler transforms to improve the convergence of certain series expansions occurring in boundary layer theory. In particular, it was found that for the problem $u_1 = u_o(\xi + \xi^3)$ for which, for large values of the Howarth-Blasius variable $\xi$, the (non-dimensional) displacement thickness is of the form $A/\xi$, such techniques could be used to predict the value of A to within 0.3% of the computed value. A similar analysis, carried out for the flow $u_1 = u_o(1+\xi)$ did help to produce an improvement in convergence for the corresponding series for the displacement thickness though not to the same degree. This type of analysis is found in many areas of boundary layer theory and a notable exponent of the art is Van Dyke (cf ref. (27)).

Chapter 3 dealt with the attempts to produce new values for the tabulated functions $F_o(\lambda)$, $G_o(\lambda)$, $F_1(\lambda)$ and $G_1(\lambda)$. The set of parameters $\left\{ \lambda, \mu, L, l^2, H \right\}$ was calculated for seven solutions covering a range of $\lambda$ from $-0.117 \leq \lambda \leq 0.14$. From each solution several points were chosen, totalling sixty-two in all, to cover the above range of $\lambda$. Polynomial models in $\lambda$ were then taken for the functions $F_o(\lambda)$, $G_o(\lambda)$, $F_1(\lambda)$ and $G_1(\lambda)$ and fitted to the data by a least squares fitting process. To check the reliability of the forms, residual quantities such as the residual sum of squares, mean modulus residual and root mean square residual were examined both over the sixty-two points used in the fitting process and at twenty-five other typical points. These quantities were found to be of approximately the same order. For small values of $\lambda$ a similar analysis was attempted using sixteen points in the range $|\lambda| \leq 0.020$ - this time, also imposing the requirement that the leading values of the coefficients should be adjusted to give a form for H continuous at $\lambda = 0$. It was found that the forms these fitting processes yielded were in good agreement for $0.015 \leq |\lambda| \leq 0.025$ with only minor smoothing required to fair both the outer and inner expansions into each other. Over the range $|\lambda| \geq 0.025$ the forms using the sixty-two points were used to calculate the tables; for $|\lambda| \leq 0.015$ the fitting for small $\lambda$ was used.

At first sight from the results given in tables 3.6, 3.7 and 3.8

it would appear that little improvement has been gained through this fitting procedure. It must however be remembered that these tables have been produced using a far larger selection of points than were the original ones and also possess the benefit that they have been adjusted to be consistent with the analytic form for H for small $\lambda$.

The fourth chapter attempts an investigation of the method in its application to compressible boundary layer problems. The conditions and properties of the Stewartson-Illingworth transformation are investigated and also an idea due to Stewartson for assessing the relative accuracies of approximate methods. This is examined for the particular compressible flow with external velocity distribution $u_1 = u_o(1-\xi)$ and the application of the methods of Howarth and Pohlhausen to this problem. The accuracy of these methods is found to be related to a condition on $\gamma$ determined from the requirement that the associated incompressible flow for this velocity distribution should attain the Falkner Skan separation profile as the leading edge Mach number becomes infinite. The accuracy of the methods was shown to be dependent on how closely this condition could be reproduced by each method. This analysis was extended to the Curle method and the condition produced both for the old and new tables was found to be in significantly better agreement with the exact condition then either of the methods of Howarth and Pohlhausen. The high accuracy to be expected from the method was confirmed by a calculation of the separation position for the compressible flow $u_1 = u_o$

$(1-\xi)$ at a leading edge Mach number of 4. An extension of the calculation to higher values of Mach number demonstrated an interesting feature of the method in the adjustment of the values of $\lambda$ and $\mu$ at separation, as against the conventional one parameter methods which assume separation to occur for a fixed value of $\lambda$.

This then summarises what has been done. The method has been investigated and perhaps slightly improved, with a new set of tables which do give a better representation to the solutions available. A compressible boundary layer problem has also been investigated and some justification given for the accuracy which has been obtained. These investigations therefore confirm the accuracy and reliability of this two-parameter method of calculation.

Appendix: Domb's Ratio Test (cf. references (4) and (5))

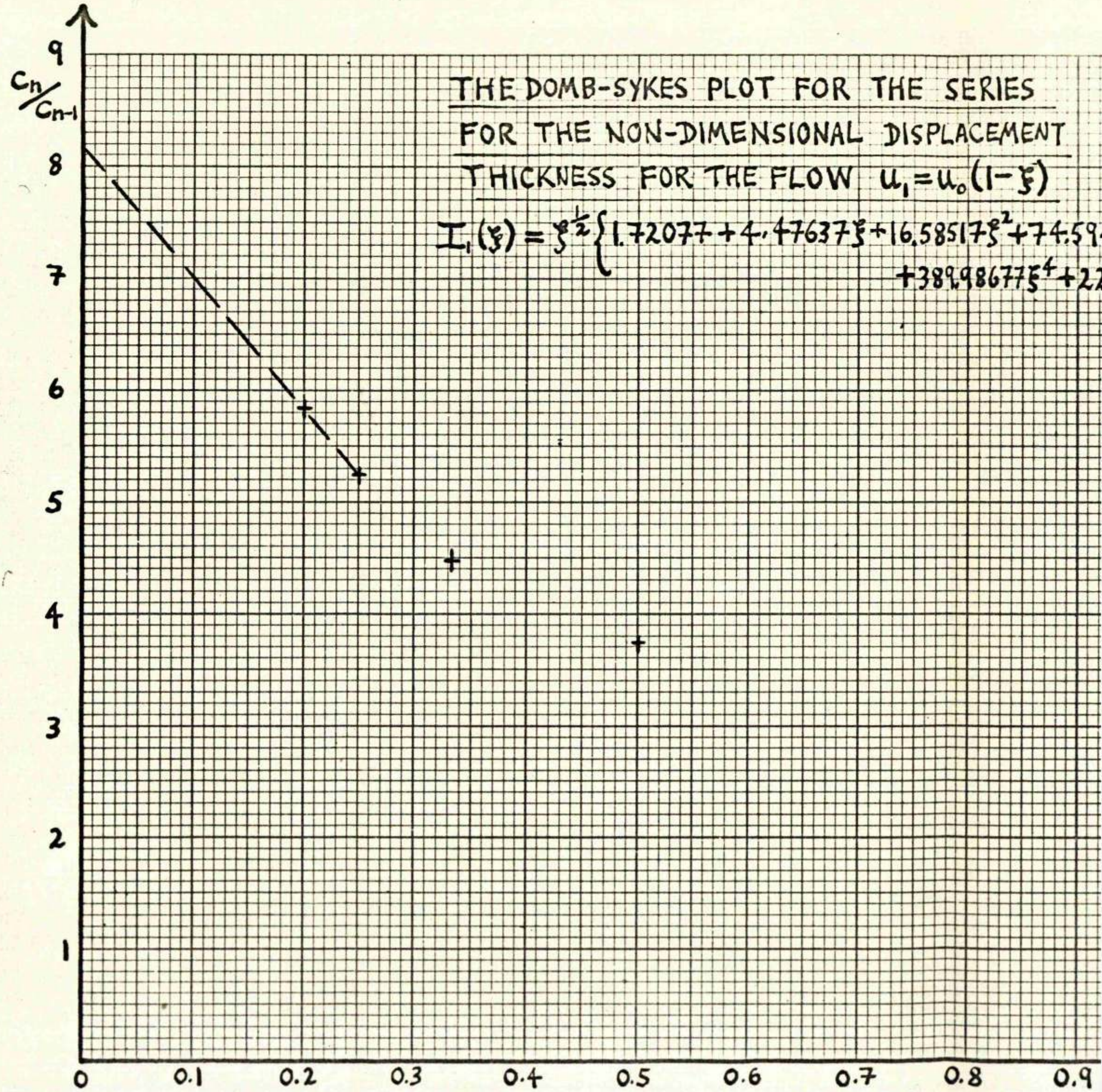
For certain common singular functions, if z is a complex variable and

$$f(Z) = \sum_{0}^{\infty} C_n Z^n = \begin{cases} C(Z_\circ \pm Z)^{\alpha}; \ \alpha \neq 0,1,\ldots\ldots \\ C(Z_\circ \pm Z)^{\alpha}\ln(Z_\circ \pm Z); \ \alpha = 0,1,\ldots \end{cases}$$

then $$C_n/C_{n-1} = \mp \frac{1}{Z_\circ}\left(1 - \frac{1+\alpha}{n}\right)$$

This relationship is useful in estimating the nearest singularity of series expansions occurring in physical problems where only the first few terms in the series are known. By plotting the ratio $C_n/C_{n-1}$ versus $\frac{1}{n}$ and extrapolating linearly to the origin the position of the nearest singularity can be estimated. Also if the plot tends to become linear then the limiting slope indicates the nature of the nearest singularity. If the signs of the $C_n$ are all positive the singularity occurs on the positive real axis. If the signs alternate the singularity occurs on the negative real axis, and by plotting $\left|C_n/C_{n-1}\right|$ versus $\frac{1}{n}$, the modulus value of the singularity may be obtained. This is known as the Domb-Sykes plot or Domb's ratio test.

Its use is illustrated for the series for the non-dimensional displacement thickness for the incompressible flow with external velocity $u_1 = u_\circ(1-\xi)$ where $\xi$ is a non-dimensional co-ordinate in the direction of the flow. Here the appropriate series is, using the symbol $I_1(\xi)$ for the non-dimensional displacement thickness,


THE DOMB-SYKES PLOT FOR THE SERIES
FOR THE NON-DIMENSIONAL DISPLACEMENT
THICKNESS FOR THE FLOW $u_1 = u_0(1-\xi)$
$I_1(\xi) = \xi^{\frac{1}{2}}\{1.72077 + 4.47637\xi + 16.58517\xi^2 + 74.59$
$+389.98677\xi^4 + 2$
$C_n/C_{n-1}$
9
8
7
6
5
4
3
2
1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

$$I_1(\xi) = 1.72077\xi^{\frac{1}{2}} + 4.47637\xi^{3/2} + 16.58517\xi^{5/2} + 74.59477\xi^{7/2} + 389.98677\xi^{9/2} + 2267.5932\xi^{11/2} + \ldots\ldots\ldots$$

$$= \xi^{\frac{1}{2}}(1.72077 + 4.47637\xi + 16.58517\xi^2 + 74.5947\xi^3 + 389.98677\xi^4 + 2267.5932\xi^5 + \ldots\ldots\ldots$$

The series inside the bracket is of the form $\sum_{0}^{\infty} c_n\xi^n$, where we now consider $\xi$ as a complex variable and we obtain the successive ratios

$C_1/C_0 = 2.601$; $C_2/C_1 = 3.705$; $C_3/C_2 = 4.498$; $C_4/C_3 = 5.228$;

$C_5/C_4 = 5.814$.

These are shown plotted versus $\frac{1}{n}$, the dotted line showing linear extrapolation on the last two points. This yields the line $I_1 = 8.16 - 11.72\frac{1}{n} = 8.16(1 - 1.43\frac{1}{n})$ which gives a value of 0.122 for the position of the singularity. The slope of the line indicates a value of $\alpha = 0.43$.

For this particular problem the position of separation is $\xi = 0.120$ and the singularity in the boundary layer equations at this point produces a square root singularity in the displacement thickness. The Domb-Sykes plot based on the first six terms of the series for the displacement thickness therefore yields a very good estimate for the position of the singularity, and hence the position of separation. The slope of the line also gives a guide as to the nature of the singularity. It would appear to be suggesting a singularity which is almost square root in nature, but since it is not unlikely that the line has not yet reached its limiting slope, a few more terms might be required to confirm this more closely.

REFERENCES

(1) BELLMAN, R., Perturbation methods applied to non-linear dynamics. J. App. Mech. 22 pp 500-502, 1955.

(2) CURLE, S. N., A two parameter method for calculating the two-dimensional incompressible laminar boundary layer. J. Roy. Aero. Soc. Vol. 71, No. 674 pp 117-123, 1967.

(3) CURLE, S. N., Accurate solutions of the laminar-boundary-layer equations, for flows having a stagnation point and separation. A.R.C. R and M No. 3164. 1960.

(4) DOMB, C., Order-disorder statistics, II A two dimensional model. Proc. Roy. Soc. A. 199 pp 199-217, 1949.

(5) DOMB, C. and SYKES, M. G., On the susceptibility of a ferromagnetic above the Curie point. Proc. Roy. Soc. A. 240 pp. 214-228, 1957.

(6) EVANS, H. L., Laminar boundary layers with uniform fluid properties, similar solutions to the velocity equations involving mass transfer. (C.P. No. 857, H.M.S.O. 1966).

(7) FRÖSSLING, N., Series developments and finite-difference calculations in the laminar boundary layer. Phys. Fluids Supplement Vol. 12, No. 12, pt. 11, pp. II-1→II-292, Dec. 1969.

(8) HARTREE, D. R., On an equation occurring in Falkner and Skan's approximate treatment of the equations of the boundary layer. Proc. Camb. Phil. Soc. 33, 223-239, 1937.

(9) HOWARTH, L., Steady flow in the boundary layer, near the surface of a cylinder in a stream. Rep. Memor. Aero. Res. Comm. Lond. No. 1632 1934 .

(10) HOWARTH, L., On the solution of the laminar boundary layer equations. Proc. Roy. Soc. A. 164, 547-579, 1938.

(11) ILLINGWORTH, C. R., Steady flow in the laminar boundary layer of a gas. Proc. Roy. Soc. A. 199, p. 533-588, 1949.

(12) LEIBENSON, L. S., The energy form of the integral condition in the theory of the boundary layer. Dokl. Ak. Nauk. S.S.S.R.2, pp. 22-24, 1935.

(13) LIBBY, P. A. and CHEN, K. K., Boundary layers with small departures from the Falkner-Skan profile. J. Fluid Mech. 33, Part 2, pp. 273-282, 1968.

(14) PICKEN, S. M., Selected points method ordinary differential equation. N.P.L. Subroutine No. P2955, 1968.

(15) POHLHAUSEN, K., Zur näherungsweisen integration der differentialgleichung der laminaren grenzschicht. Z. Angew. Math. Mech. 1, pp. 252-68, 1921.

(16) ROSENHEAD, L. (Ed), Laminar boundary layers. O.U.P. 1963.

(17) SCHLICHTING, H. and ULRICH, A., Zur berechnung des umschlages laminar/turbulent 1. Ber. Lilienthal-Ges. Luftfaurtf. 510, pp. 75-135. J.B. Dtsch. Luftfahrtf. 1, pp. 8-35 1942 .

(18) SMITH, A. M. O., A decade of boundary-layer research. App. Mechs. Rev. Vol. 23, No. 1 , pp 1-9, 1970.

(19) STEWARTSON, K., Correlated incompressible and compressible boundary layers. Proc. Roy. Soc. A. 200, p. 84-100, 1949.

(20) STEWARTSON, K., CATHERALL, D. and WILLIAMS, P. G., Viscous flow past a flat plate with uniform injection. Proc. Roy. Soc. A, Vol. 284, pp 370-396, 1965.

(21) TANI, I., On the solution of the laminar boundary layer equations. J. Phys. Soc. Japan, Vol. IV, p. 149-154, 1949.

(22) TERRILL, R. M., Laminar boundary-layer flow near separation with and without suction. Phil. Trans. A. 253, pp. 55-100, 1960

(23) THWAITES, B., Approximate calculation of the laminar boundary layer. Aeronaut. Quart. 1, pp 245-280, 1949.

(24) THWAITES, B., The development of laminar boundary layers under continuous suction. Part II - approximate methods of solution. A.R.C. 12,699 , 1949.

(25) TIFFORD, A. N., Heat transfer and frictional effects in laminar boundary layers. Part 4, universal series solutions. Tech. Rep. Wright Air Dev. Center No. 53-288, 1954.

(26) TRUCKENBRODT, E., Ein quadraturrerfahren zur berechnung der laminaren und turbulenten reibungsschichten bei ebener und rotationssymmetrischer stromung. Ingenieur Archiv., Vol. 20, pp 211-228, 1952.

(27) VAN DYKE, M., Higher approximations in boundary-layer theory. Part 3. Parabola in uniform stream. J. Fluid Mech. 19, pp 145-159, 1964.

(28) WALZ, A., Ein neuer ansatz für das geschwindigkeitsprofil der laminaren reibungsschicht. Ber. Lilienthal-Ges. Luftfahrtf, No. 141, pp. 8-12, 1941.

(29) SYSTEM/360 Scientific subroutine package (360A-CM-03X) Version 111, Programmer's Manual, FORM NO. H20-0205-3.